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Représentations modulaires et structure locale des groupes finis

Modular representations and local structure of finite groups

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Résumé

Cette thèse s'inscrit dans la recherche d'une preuve modulaire du Z_p^*-théorème pour p impair, dont la seule démonstration connue repose sur la classification des groupes finis simples. Soit $\mathcal{O}$ une extension assez grande de l'anneau p-adique $\mathbb{Z}_p$, et k son corps résiduel. Soit G un groupe fini, e un bloc de l'algèbre $\mathcal{O}G$ et $H = C_G(P)$ le centralisateur d'un p-sous-groupe de G. Si le sous-groupe H contrôle la fusion du bloc e en un sens très fort, nous prouvons l'existence d'une équivalence stable de type Morita entre le bloc e et un bloc f de l'algèbre $\mathcal{O}H$, sous réserve qu'un groupe de défaut du bloc e soit abélien ou que son centre ne soit pas cyclique. Nous étendons ainsi un résultat déjà connu pour le bloc principal.

Pour construire le bimodule qui induit cette équivalence stable, nous sommes amené à étudier les modules sur une algèbre de bloc $\mathcal{O}Ge$ qui possèdent une source d'endopermutation fusion-stable, et que nous appelons des modules «Brauer-compatibles». Nous montrons en particulier comment la construction «slash» de Dade peut être appliquée à ces modules, et comment cette construction peut être rendue fonctorielle si on la restreint à une sous-catégorie «Brauer-compatible» de la catégorie des $\mathcal{O}Ge$-modules. Nous prouvons qu'un $\mathcal{O}Ge$-module indécomposable Brauer-compatible est caractérisé par une sous-paire vortex (Q, e_Q), un module source V, et un module projectif indécomposable sur l'algèbre de bloc locale $k[N_G(Q, e_Q)/Q]\bar{e}_Q$ associée à la sous-paire (Q, e_Q). Nous donnons ainsi une formulation fonctorielle de la correspondance de Puig pour les modules Brauer-compatibles.

Abstract

This thesis is related to the pursuit of a modular proof of the odd Z_p^*-theorem, while the only known proof of that theorem relies on the classification of finite simple groups. Let $\mathcal{O}$ be a big enough extension of the p-adic ring $\mathbb{Z}_p$, and k be its residue field. Let G be a finite group, e be a block of the algebra $\mathcal{O}G$, and $H = C_G(P)$ be the centraliser of a p-subgroup of G. If the subgroup H controls the fusion of the block e in a very strong sense, we prove the existence of a stable equivalence of Morita type between the block e and a block f of the algebra $\mathcal{O}H$, provided that a defect group of the block e is abelian or has a noncyclic center. This extends a result previously known for the principal block.

In order to construct the bimodule that induces this stable equivalence, we are led to study the class of modules over a block algebra $\mathcal{O}Ge$ that admit a fusion-stable endopermutation source. We call them “Brauer-friendly” modules. In particular, we show that Dade’s “slash” construction applies to these modules, and that this construction can be turned into a functor over a “Brauer-friendly” subcategory of the category of $\mathcal{O}Ge$-modules. We prove that an indecomposable Brauer-friendly module is characterised by a vertex subpair (Q, e_Q), a source module V, and a projective indecomposable module over the local block algebra $k[N_G(Q, e_Q)/Q]\bar{e}_Q$ attached to the subpair (Q, e_Q). This provides a functorial version of the Puig correspondence for Brauer-friendly modules.

Dans la cabine spatiale, le centralisateur dicte ses ordres depuis le premier jour. A leur cinq-cent quatrième jour de vol, les servants s'inquiètent et s'insurgent comme jadis sur le Bounty :

« La fusion est-elle sous contrôle ? Ne menace-t-elle pas le vortex par ses modules projectifs ? La fusion stable peut-elle calmer nos inquiétudes comme si, sous la fusion, la plage ? Sa stabilité dans le temps et sous les latitudes extrêmes est-elle prouvée ? Que faire des déchets ? Le groupe ou le sous-groupe est-il vraiment fini, ne peut-on encore s'en servir, réduisant l'empreinte carbone ?

Non à l'endopermutation ! Oui à l'endomutation qui refuse le libre-échange comme le troc, la monnaie. Quand nous endomuterons, alors la fusion s'opérera dans la non-violence, sans déchet ni carbone, et nous atteindrons sans encombre l'étoile α-$H = C_G(P)$ *! »*

Georges Curinier, 14 février 2013

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Introduction

Ce travail s'inscrit dans le cadre de la théorie des représentations, et plus précisément dans la branche de cette théorie qui traite des représentations des groupes finis en caractéristique positive, appelées représentations modulaires. Au cours des trente dernières années, l'étude des représentations modulaires a connu des développements qui reposent sur l'interaction entre des outils issus de la théorie des catégories (équivalences de Morita, équivalences dérivées, équivalences stables), mais aussi de l'étude locale des groupes finis (systèmes de fusion). Cette interaction peut se révéler très féconde ; elle nous a permis d'obtenir le résultat principal de cette thèse.

La présente introduction se divise en trois parties. La première partie est un survol de l'état des connaissances dans les domaines mathématiques concernés par notre travail (pour plus de détails et des références à la littérature, le lecteur pourra se reporter au premier chapitre qui vient ensuite). La seconde partie retrace le travail qui a mené à cette thèse de doctorat. La dernière partie, en anglais, en condense les principaux résultats.

Cadre théorique

Équivalences catégoriques entre algèbres de blocs

Fixons un nombre premier p. Un système p-modulaire est un triplet $(\mathbb{K}, \mathcal{O}, k)$, où $\mathcal{O}$ est un anneau de valuation discrète complet, $\mathbb{K}$ son corps des fractions de caractéristique nulle, et k son corps résiduel de caractéristique p. Pour éviter les problèmes de rationalité, nous supposerons que le corps $\mathbb{K}$ est toujours «suffisamment gros».

Soit G un groupe fini, et RG son algèbre de groupe sur l'anneau R, qui peut être $\mathbb{K}$, $\mathcal{O}$ ou k. Une représentation du groupe G sur l'anneau R est un RG-module, que nous supposerons toujours de type fini sur l'anneau R. La théorie des représentations cherche à décrire la catégorie ${}_{RG}\mathbf{Mod}$ des RG-modules et morphismes de RG-modules. Un bloc de l'algèbre RG est un idempotent primitif du centre $Z(RG)$. Pour $R = \mathbb{K}$, ces blocs sont en correspondance bijective avec les caractères ordinaires irréductibles du groupe G. Ce cas est très bien connu, aussi nous ne l'aborderons pas dans cette thèse. Pour $R = \mathcal{O}$ ou $R = k$, la structure des blocs est beaucoup plus riche. La réduction modulo l'idéal maximal $\mathfrak{m}$ de l'anneau $\mathcal{O}$ induit une correspondance bijective entre les blocs de l'algèbre $\mathcal{O}G$ et ceux de l'algèbre kG. Autant que possible, nous travaillerons donc sur l'anneau local $\mathcal{O}$.

Les blocs $e_1, \ldots, e_n$ de l'algèbre de groupe $\mathcal{O}G$ correspondent à une décomposition $\mathcal{O}G \simeq \mathcal{O}Ge_1 \times \cdots \times \mathcal{O}Ge_n$ de l'algèbre $\mathcal{O}G$ en un produit direct d'algèbres indécomposables. Cette décomposition entraîne une décomposition de la catégorie des $\mathcal{O}G$-modules sous la forme

$$ {}_{\mathcal{O}G}\mathbf{Mod} \;\simeq\; {}_{\mathcal{O}Ge_1}\mathbf{Mod} \oplus \cdots \oplus {}_{\mathcal{O}Ge_n}\mathbf{Mod}. $$

Dorénavant, nous fixerons un bloc e et nous étudierons uniquement les $\mathcal{O}Ge$-modules.

Le résultat principal que nous visons dans cette thèse est une équivalence de nature catégorique entre deux blocs de groupes finis. Considérons donc G, H deux groupes finis, et e, f des blocs respectifs des algèbres $\mathcal{O}G$, $\mathcal{O}H$. Il existe trois types principaux d'équivalences purement catégoriques entre les algèbres $\mathcal{O}Ge$ et $\mathcal{O}Hf$, que nous énumérons ci-dessous du plus fort au moins fort.

- l'équivalence de Morita, lorsque les catégories de modules ${}_{\mathcal{O}Ge}\mathbf{Mod}$ et ${}_{\mathcal{O}Hf}\mathbf{Mod}$ sont équivalentes ;

- l'équivalence dérivée, lorsque les catégories dérivées bornées $\mathbf{D^b}({}_{\mathcal{O}Ge}\mathbf{Mod})$ et $\mathbf{D^b}({}_{\mathcal{O}Hf}\mathbf{Mod})$ sont équivalentes en tant que catégories triangulées. La catégorie dérivée $\mathbf{D^b}({}_{\mathcal{O}Ge}\mathbf{Mod})$ est obtenue en considérant la catégorie des complexes de $\mathcal{O}Ge$-modules cohomologiquement bornés, et en rendant inversibles les morphismes de cette catégorie qui induisent des isomorphismes en cohomologie (les quasi-isomorphismes).

- l'équivalence stable, lorsque les catégories stables $\mathbf{Stab}({}_{\mathcal{O}Ge}\mathbf{Mod})$ et

$\mathbf{Stab}({}_{\mathcal{O}Hf}\mathbf{Mod})$ sont équivalentes en tant que catégories triangulées. La catégorie stable $\mathbf{Stab}({}_{\mathcal{O}Ge}\mathbf{Mod})$ est obtenue comme un quotient de la catégorie des $\mathcal{O}Ge$-modules, en annulant formellement les objets projectifs de cette catégorie.

Un célèbre théorème, dû à Morita, affirme qu'une équivalence de catégories ${}_{\mathcal{O}Hf}\mathbf{Mod} \to {}_{\mathcal{O}Ge}\mathbf{Mod}$ est nécessairement, à isomorphisme près, un foncteur du type $M \otimes_{\mathcal{O}Hf} -$, où M est un $(\mathcal{O}Ge, \mathcal{O}Hf)$-bimodule possédant certaines «bonnes» propriétés. Ce théorème a été généralisé par Rickard aux équivalences entre catégories dérivées. Bien qu'il n'admette pas de généralisation connue aux équivalences entre catégories stables, on se focalise généralement sur les équivalences stables «de type Morita», c'est-à-dire définies grâce au produit tensoriel par un certain bimodule M, comme ci-dessus.

Notre recherche adopte donc ce point de vue apparemment paradoxal : nous étudions des équivalences entre catégories de modules, ce qui semble indiquer que les modules, individuellement, nous intéressent peu ; mais ces équivalences catégoriques sont définies par des bimodules, c'est-à-dire des modules d'un type particulier, qui sont finalement les objets principaux de notre travail.

Modules de permutation et d'endopermutation

Quels sont les types de modules qui vont nous intéresser ? Considérons un groupe fini G et un $\mathcal{O}G$-module M. On dit que M est un $\mathcal{O}G$-module de permutation si c'est un $\mathcal{O}$-module libre qui admet une base stable par l'action du groupe G. On dit que M est un $\mathcal{O}G$-module de p-permutation s'il est isomorphe à un facteur direct d'un $\mathcal{O}G$-module de permutation. La catégorie ${}_{\mathcal{O}G}\mathbf{Perm}$ des $\mathcal{O}G$-modules de p-permutation est une sous-catégorie de ${}_{\mathcal{O}G}\mathbf{Mod}$ qui est stable par facteurs directs et par sommes directes. Les modules de p-permutation se comportent aussi très bien vis-à-vis de l'induction et de la restriction.

Par ailleurs, l'algèbre d'endomorphismes $\mathrm{End}_{\mathcal{O}}(M)$ est munie d'une action du groupe G par automorphismes d'algèbres. C'est donc ce qu'on appelle une G-algèbre, qu'on peut aussi considérer comme un $\mathcal{O}G$-module. On dit que M est un $\mathcal{O}G$-module d'endopermutation si l'algèbre $\mathrm{End}_{\mathcal{O}}(M)$ est un $\mathcal{O}G$-module de permutation, et que M est un $\mathcal{O}G$-module d'endo-p-permutation si l'algèbre $\mathrm{End}_{\mathcal{O}}(M)$ est un $\mathcal{O}G$-module de p-permutation. La catégorie des $\mathcal{O}G$-modules d'endo-p-permutation n'est pas un objet très intéressant, car elle n'est généralement pas stable par sommes directes. On dit que deux $\mathcal{O}G$-modules d'endo-p-

permutation M et N sont compatibles si la somme directe $M \oplus N$ est encore un module d'endo-p-permutation. Pour la même raison, les modules d'endo-p-permutation se comportent mal vis-à-vis de l'induction si l'on n'impose pas des conditions restrictives.

Si P est un p-groupe, alors tout $\mathcal{O}P$-module de p-permutation ou d'endo-p-permutation, est en fait un module de permutation ou d'endopermutation, respectivement. On sait classifier depuis longtemps les $\mathcal{O}P$-modules de permutation. Plus récemment, on a obtenu une classification de tous les $\mathcal{O}P$-modules d'endopermutation. Ces objets jouent un rôle important dans la théorie des représentations modulaires. Pour paraphraser Dade, la famille des modules d'endopermutation est à la fois suffisamment petite pour en permettre la classification et suffisamment large pour être utilisable.

Le foncteur de Brauer et la construction slash

Soit P un p-groupe fixé. Si X est un ensemble muni d'une action du groupe P, on note X^P le sous-ensemble des points de X fixés par P. Soit $\mathcal{O}X$ le $\mathcal{O}P$-module de permutation associé à X. On dispose d'une application naturelle $\mathrm{br}_P : (\mathcal{O}X)^P \to k(X^P)$, définie par la troncature et la réduction modulo l'idéal maximal $\mathfrak{m}$ de l'anneau local $\mathcal{O}$. L'application br_P est même un morphisme d'algèbres si X est un groupe muni d'une action de P par automorphismes de groupes. Le foncteur de Brauer

$$\mathrm{Br}_P : {}_{\mathcal{O}P}\mathbf{Mod} \to {}_{k}\mathbf{Mod}$$

est un foncteur additif qui généralise la construction précédente à un $\mathcal{O}P$-module M quelconque. Si $M = \mathcal{O}X$ est un $\mathcal{O}P$-module de permutation, on dispose d'un isomorphisme naturel $\mathrm{Br}_P(M) \simeq k(X^P)$. Dans le cas général, le k-espace vectoriel $\mathrm{Br}_P(M)$ apparaît comme un quotient du module de points fixes M^P, et la projection $\mathrm{br}_P^M : M^P \to \mathrm{Br}_P(M)$ est appelée le morphisme de Brauer. Par convention, nous notons ce morphisme avec un b minuscule, pour le différencier du foncteur de Brauer qui est doté d'un B majuscule.

Si M est un $\mathcal{O}G$-module et P un p-sous-groupe du groupe fini G, alors l'application br_P^M induit une structure naturelle de $kN_G(P)$-module sur le quotient de Brauer $\mathrm{Br}_P(M)$. Si A est une $\mathcal{O}$-algèbre sur laquelle le groupe P agit par automorphismes d'algèbres, alors l'application br_P^A induit une structure naturelle de k-algèbre sur le quotient de Brauer $\mathrm{Br}_P(A)$.

Par construction, le foncteur de Brauer a de bonnes propriétés, notamment de transitivité, lorsqu'on l'applique à un module de permutation ou de p-permutation. Mais il se comporte assez mal vis-à-vis d'un module M quelconque. Il est alors préférable de travailler avec l'algèbre d'endomorphismes $\mathrm{End}_{\mathcal{O}}(M)$.

En particulier, si M est un $\mathcal{O}P$-module d'endopermutation, alors le quotient de Brauer $\mathrm{Br}_P(\mathrm{End}_{\mathcal{O}}(M))$ est isomorphe à une algèbre de matrices sur le corps k. En d'autres termes, il existe un k-espace vectoriel $M(P)$ tel que

$$\mathrm{Br}_P(\mathrm{End}_{\mathcal{O}}(M)) \simeq \mathrm{End}_k(M(P)).$$

La correspondance $M \mapsto M(P)$ ainsi définie s'appelle la construction slash, ou déflation-restriction. Si M est un $\mathcal{O}G$-module et P un p-sous-groupe de G tel que la restriction $\mathrm{Res}^G_P M$ est un module d'endopermutation, alors le k-espace vectoriel $M(P)$ admet une structure de $kN_G(P)$-module issue de l'action du normalisateur $N_G(P)$ sur l'algèbre $\mathrm{End}_{\mathcal{O}P}(M)$. Cette structure n'est cependant pas unique ; elle peut être «tordue» par un caractère linéaire du groupe $N_G(P)/PC_G(P)$. Surtout, la construction slash est *a priori* fonctorielle au niveau des algèbres d'endomorphismes, mais pas des modules, ce qui pose des difficultés dans son utilisation.

Structure locale des groupes finis et de leurs blocs

Le nombre premier p étant toujours fixé, on appelle structure p-locale d'un groupe fini G l'action du groupe G par conjugaison sur l'ensemble ordonné de ses p-sous-groupes. Si l'on s'intéresse à un bloc e de l'algèbre de groupe $\mathcal{O}G$, on est amené à remplacer les p-sous-groupes de G par des objets plus précis, les sous-paires, dont nous allons esquisser la construction.

Si P est un p-sous-groupe du groupe G, le morphisme de Brauer relatif à la P-algèbre $\mathcal{O}G$ apparaît naturellement comme un morphisme d'algèbres

$$\mathrm{br}_P : \mathcal{O}G \to kC_G(P).$$

Une e-sous-paire du groupe G est alors un couple (P, e_P), où P est un p-sous-groupe de G et e_P est un bloc de l'algèbre $\mathcal{O}C_G(P)$ dont la réduction $\bar{e}_P \in kC_G(P)$ appartient à la sous-algèbre $kC_G(P)\,\mathrm{br}_P(e)$. Le groupe G agit par conjugaison sur l'ensemble de ses e-sous-paires, lequel est muni par ailleurs d'un ordre qui raffine l'inclusion des p-sous-groupes de G. Cette information est ce qu'on appelle la structure locale du bloc e. Elle est encodée dans une catégorie, la catégorie de Brauer $\mathbf{Br}(G, e)$, dont les objets sont les e-sous-paires de G et dont les flèches sont induites par la conjugaison dans G.

Les théorèmes classiques de Sylow se généralisent aux e-sous-paires. Ainsi, les e-sous-paires maximales forment une orbite sous l'action du groupe G par conjugaison. Un groupe de défaut du bloc e est un p-sous-groupe D de G tel qu'il existe une e-sous-paire maximale de la forme (D, e_D). Si l'on fixe un telle sous-paire maximale (D, e_D), et un p-sous-groupe P du groupe de défaut D, il existe une unique sous-paire de la forme (P, e_P) qui est contenue dans la sous-paire (D, e_D).

Nous avons parlé plus tôt de divers types d'équivalences entre algèbres de blocs. Au delà de sa nature purement catégorique, on espère généralement qu'une telle équivalence possède une certaine compatibilité avec les structures locales des blocs considérés. Nous allons maintenant préciser ce que peut signifier «compatibilité» dans la phrase précédente.

Considérons deux groupes G et H qui possèdent un p-sous-groupe D en commun. Soient e et f des blocs respectifs des algèbres de groupes $\mathcal{O}G$ et $\mathcal{O}H$. Supposons que ces deux blocs admettent le p-sous-groupe D comme groupe de défaut, et fixons des sous-paires maximales (D, e_D) et (D, f_D), relatives aux blocs e et f, respectivement. Pour tout sous-groupe P de D, notons (P, e_P) et (P, f_P) les sous-paires respectives de (D, e_D) et (D, f_D) associées au p-sous-groupes P. Si la correspondance bijective $(P, e_P) \leftrightarrow (P, f_P)$ induit une équivalence entre les catégories de Brauer $\mathbf{Br}(G, e)$ et $\mathbf{Br}(H, f)$, on dit que les blocs e et f ont la même structure locale.

Avec ces hypothèses, soit M un $(\mathcal{O}Ge, \mathcal{O}Hf)$-bimodule qui établit une équivalence de Morita, ou simplement une équivalence stable, entre les algèbres de blocs $\mathcal{O}Ge$ et $\mathcal{O}Hf$. On s'attend à obtenir, pour tout sous-groupe P de D, un bimodule M_P qui induit une équivalence de Morita entre les algèbres de blocs «locales» $\mathcal{O}N_G(P, e_P)e_P$ et $\mathcal{O}N_H(P, f_P)f_P$. La construction du bimodule M_P, à l'aide du foncteur de Brauer Br_P, est bien connue si M est un $\mathcal{O}(G \times H)$-module de p-permutation. Elle se généralise même à un complexe de bimodules de p-permutation qui induit une équivalence dérivée entre algèbres de blocs; on parle alors d'équivalence «splendides». Des outils plus généraux de localisation des bimodules existent. Ils font appel au langage des algèbres sources et définissent, par exemple, des équivalences de Puig.

Réciproquement, si on dispose d'une famille de bimodules «locaux» qui induisent des équivalences de Morita, on s'attend à pouvoir construire, par recollement, un $(\mathcal{O}Ge, \mathcal{O}Hf)$-bimodule qui établit une équivalence stable de type Morita entre les algèbres de blocs $\mathcal{O}Ge$ et $\mathcal{O}Hf$. En plus d'outils pour localiser, une telle construction nécessite des outils pour construire un tel bimodules «global».

Construction de modules indécomposables

Un $(\mathcal{O}Ge, \mathcal{O}Hf)$-bimodule peut aussi être considéré comme un $\mathcal{O}(G\times H)(e\otimes f^{\mathrm{o}})$-module, où f^{o} est l'image du bloc f par l'isomorphisme naturel entre l'algèbre de groupe $\mathcal{O}H$ et son algèbre opposée $(\mathcal{O}H)^{\mathrm{op}}$. Si l'on souhaite que ce bimodule induise une équivalence de Morita ou une équivalence stable, il est nécessairement indécomposable.

Nous pouvons donc, dans un premier temps, considérer un seul groupe fini G (qui joue le rôle du produit direct $G \times H$), choisir un bloc e de l'algèbre de groupe $\mathcal{O}G$, et étudier les propriétés d'un $\mathcal{O}Ge$-module indécomposable M. La théorie de Green associe à ce module indécomposable un p-sous-groupe P de G appelé vortex de M et un $\mathcal{O}P$-module indécomposable V appelé source de M. Le couple (P, V) est bien défini, à conjugaison près dans le groupe G. On associe également à M un $\mathcal{O}N_G(P)$-module indécomposable M' de vortex P et de source V, qu'on appelle le correspondant de Green de M, et qui est bien défini à isomorphisme près. La correspondance $M \leftrightarrow M'$ est bijective, mais elle a l'inconvénient de ne pas être fonctorielle.

Un cas très favorable est celui où M est un $\mathcal{O}G$-module de p-permutation, c'est-à-dire où la source V est le $\mathcal{O}P$-module trivial $\mathcal{O}$. Dans ce cas, le quotient de Brauer $\mathrm{Br}_P(M)$ est isomorphe à la réduction $k \otimes_{\mathcal{O}} M'$ du correspondant de Green M'. Le module indécomposable M est donc caractérisé de façon unique par le triplet $(P, \mathcal{O}, M(P))$, où le p-groupe P est un vortex de M, le $\mathcal{O}P$-module trivial $\mathcal{O}$ est sa source, et le $kN_G(P)/P$-module projectif indécomposable $M(P) = \mathrm{Br}_P(M)$ est son quotient de Brauer relativement au p-sous-groupe P. En particulier, ce troisième invariant dépend fonctoriellement du module M, ce qui le rend plus facile à manier que le correspondant de Green.

Plus généralement, la correspondance de Puig permet de caractériser un $\mathcal{O}G$-module indécomposable par un triplet $(P, V, \bar{M})$, où P est un vortex de M, V une source de M, et $\bar{M}$ un certain $kN_G(P,V)/P$-module projectif indécomposable.

Pour construire, par recollement, un $(\mathcal{O}Ge, \mathcal{O}Hf)$-bimodule M qui a vocation à induire une équivalence stable entre les algèbres de blocs $\mathcal{O}Ge$ et $\mathcal{O}Hf$, on s'attend donc à procéder en trois temps. On identifie d'abord un vortex de M, qui devrait être le sous-groupe diagonal ΔD, où D est le groupe de défaut commun aux blocs e et f. On construit ensuite sa source V, qui devrait être un $\mathcal{O}\Delta D$-module d'endopermutation. Pour cela, on dispose d'outils de recollement développés par Puig, Bouc et Thévenaz. Enfin, on doit caractériser le troisième invariant $\bar{M}$ et comprendre précisément ses liens avec le bimodule M.

Le Z_p^*-théorème

Le 20[e] siècle a été le cadre d'une longue marche vers la classification des groupes finis simples, qui s'est achevée au début des années 1980. Dans ce contexte, en 1966, Glauberman a démontré un résultat connu sous le nom de Z^*-théorème, qui porte sur la structure 2-locale des groupes finis.

Soit G un groupe fini et H un sous-groupe de G. On dit que H contrôle la p-fusion dans le groupe G si les groupes G et H ont la même structure p-locale. Ceci revient à demander que les catégories de Brauer $\mathbf{Br}(G, e_0)$ et $\mathbf{Br}(H, f_0)$ soient équivalentes, où e_0 et f_0 sont des blocs particuliers des algèbres $\mathcal{O}G$ et $\mathcal{O}H$, les blocs principaux. Par exemple, s'il existe un sous-groupe S distingué dans G, dont l'ordre n'est pas un multiple de p (on dit que S est un p'-groupe), et tel que $G = SH$, alors le sous-groupe H contrôle la p-fusion dans G. Le Z^*-théorème établit, dans le cas $p = 2$, une réciproque partielle de ce résultat.

$\mathbf{Z}^*$-théorème. *Si P est un 2-sous-groupe de G dont le centralisateur $C_G(P)$ contrôle la 2-fusion dans le groupe G, alors il existe un $2'$-sous-groupe S distingué dans G tel que $G = SC_G(P)$.*

Ce cas particulier $H = C_G(P)$ est intéressant, car la factorisation $G = SC_G(P)$ peut être caractérisée à l'aide de la théorie des représentations modulaires. Comme nous le montrerons, le groupe G admet une telle factorisation si, et seulement si, le foncteur de restriction $\mathrm{Res}^G_{C_G(P)}$ induit une équivalence entre les catégories de modules ${}_{\mathcal{O}Ge_0}\mathbf{Mod}$ et ${}_{\mathcal{O}Hf_0}\mathbf{Mod}$, ce qui revient à dire que la restriction $\mathrm{Res}^{G\times G}_{G\times H}\,\mathcal{O}Ge_0$ est un $(\mathcal{O}Ge_0, \mathcal{O}Hf_0)$-bimodule qui induit une équivalence de Morita entre les algèbres de blocs principaux $\mathcal{O}Ge_0$ et $\mathcal{O}Hf_0$.

C'est justement la théorie des représentations modulaires qui a permis à Glauberman de prouver son Z^*-théorème, qui a ensuite joué un rôle important dans la classification des groupes finis simples. Grâce à cette classification, on a pu prouver que l'énoncé du Z^*-théorème ci-dessus reste vrai si l'on remplace le nombre 2 par un nombre premier p impair ; cette généralisation a été baptisée le Z_p^*-théorème. Depuis lors, on espère obtenir une preuve de ce Z_p^*-théorème, pour p impair, par des méthodes issues des représentations modulaires, et surtout indépendantes de la classification des groupes finis simples. Les efforts en ce sens, menés notamment par Robinson, sont pour l'instant restés vains. Ils ont toutefois amené un certain nombre de résultats, qui sont à l'origine de notre travail de thèse, et que nous allons maintenant évoquer.

Fixons à présent un nombre premier p impair, et plaçons nous dans l'optique d'une preuve par induction du Z_p^*-théorème. Pour cela, on considère un groupe fini G, muni d'un p-sous-groupe P, qui est un contre-exemple minimal à ce théorème. Le centralisateur $H = C_G(P)$ contrôle la p-fusion dans G, mais le groupe G n'admet pas de factorisation sous la forme $G = SC_G(P)$ pour un p'-sous-groupe S distingué dans G. Par contre, une telle factorisation existe pour tout sous-groupe local de G. Dans ce cadre, Broué a démontré que le foncteur de restriction Res_H^G induit une équivalence stable entre les algèbres de blocs principaux $\mathcal{O}Ge_0$ et $\mathcal{O}Hf_0$.

Plaçons-nous maintenant dans le cas d'un groupe fini G qui admet une factorisation du type $G = SH$, où $H = C_G(P)$ est le centralisateur d'un p-sous-groupe, et S est un p'-sous-groupe distingué de G. Comme nous l'avons déjà dit, il existe alors une équivalence de Morita entre les algèbres de blocs principaux $\mathcal{O}Ge_0$ et $\mathcal{O}Hf_0$. Plus généralement, Külshammer et Robinson ont montré qu'il existe une équivalence de Morita entre les algèbres de blocs $\mathcal{O}Ge$ et $\mathcal{O}Hf$, dès lors que e est un bloc de l'algèbre $\mathcal{O}G$ tel que $\mathrm{br}_P(e) \neq 0$, et que f est l'unique bloc de l'algèbre $\mathcal{O}H$ tel que $\mathrm{br}_P(e) = \bar{f} \in kH$.

Revenons à la situation d'un contre-exemple minimal au Z_p^*-théorème décrite ci-dessus, en fixant un bloc non principal e de $\mathcal{O}G$ et son correspondant f dans $\mathcal{O}H$. Nous disposons alors d'une famille d'équivalences de Morita locales, et nous nous attendons à pouvoir les recoller pour obtenir une équivalence stable entre les algèbres de blocs $\mathcal{O}Ge$ et $\mathcal{O}Hf$. L'existence de cette équivalence stable se laisse deviner dans certains travaux de Robinson en rapport avec le Z_p^*-théorème, et c'est elle qui a motivé notre travail de thèse.

Organisation de la thèse et principaux résultats

Dans la première partie de cette introduction, nous avons brossé, à grands traits, le paysage mathématique dans lequel nous avons évolué au cours de notre doctorat. Nous allons maintenant retracer, de façon plus précise, le chemin que nous y avons parcouru, et les quelques cailloux que nous avons laissés derrière nous.

Équivalences de Morita et algèbres fortement graduées

Notre premier objectif était de comprendre les travaux de Robinson en lien avec le Z_p^*-théorème pour p impair, et si possible de les prolonger pour tendre vers une preuve modulaire de ce théorème. Dans cet esprit, nous avons étudié entre autres deux articles, [Rb-1986] et [KR-1986], qui portent sur les propriétés d'un groupe G muni d'un p-sous-groupe P et d'un p'-sous-groupe S distingué dans G, tels que $G = SC_G(P)$.

Cette situation où un groupe G possède un sous-groupe distingué S est un classique de la théorie des représentations, étudié par ce que certains appellent la théorie de Clifford. Fixons un bloc b de l'algèbre $\mathcal{O}S$, notons $b' \in \mathcal{O}S$ la somme des conjugués de b sous l'action du groupe G, et G_b le stabilisateur de b dans G. La catégorie ${}_{\mathcal{O}Gb'}\mathbf{Mod}$ des représentations du groupe G «au dessus» du bloc b est équivalente à la catégorie ${}_{\mathcal{O}G_bb}\mathbf{Mod}$. L'algèbre $\mathcal{O}G_bb$ possède une graduation naturelle par le groupe quotient $\bar{G}_b = G_b/S$, qui en fait une algèbre fortement $\bar{G}_b$-graduée.

Plaçons-nous dans le cas où S est un p'-groupe normal tel que $G = SC_G(P)$. Si le p-groupe P fixe le bloc b, alors il existe un unique bloc b_P de l'algèbre $\mathcal{O}C_S(P)$ tel que $\mathrm{br}_P(b) = \bar{b}_P$ dans l'algèbre $kC_S(P)$. Il s'agit là d'un cas particulier de la correspondance de Glauberman. Nous pouvons alors redire pour le bloc b_P ce que nous venons de dire pour le bloc b. Le stabilisateur du bloc b_P dans le groupe $C_G(P)$ est le centralisateur $C_{G_b}(P)$, et il y a un isomorphisme naturel $C_{G_b}(P)/C_S(P) \simeq \bar{G}_b$. Notons $b'_P \in \mathcal{O}C_S(P)$ la somme des conjugués de b_P sous l'action du groupe $C_G(P)$. Les catégories ${}_{\mathcal{O}C_G(P)b'_P}\mathbf{Mod}$ et ${}_{\mathcal{O}C_{G_b}(P)b_P}\mathbf{Mod}$

sont équivalentes, et l'algèbre $\mathcal{O}C_{G_b}(P)b_P$ est fortement $\bar{G}_b$-graduée.

Külshammer et Robinson démontrent que les algèbres $\mathcal{O}G_b b$ et $\mathcal{O}C_{G_b}(P)b_P$ sont équivalentes au sens de Morita. Pour cela, ils utilisent de façon importante leurs structures d'algèbres fortement $\bar{G}_b$-graduées : ils leur associent des invariants cohomologiques, à savoir des éléments du groupe de cohomologie $H^2(\bar{G}_b, \mathcal{O}^\times)$, dont l'égalité est la clé de leur preuve. L'équivalence de Morita $\mathcal{O}G_b b \sim \mathcal{O}C_{G_b}(P)b_P$ amène ensuite une équivalence de Morita $\mathcal{O}Gb' \sim \mathcal{O}C_G(P)b'_P$, puis des équivalences de Morita $\mathcal{O}Ge \sim \mathcal{O}C_G(P)e_P$, où e et e_P sont des blocs des algèbres $\mathcal{O}Gb'$ et $\mathcal{O}C_G(P)b'_P$, respectivement.

Rappelons que, dans l'optique d'un contre-exemple minimal au Z_p^*-théorème, la factorisation $G = SC_G(P)$ apparaît au niveau «local». Il est alors important de très bien comprendre ces équivalences de Morita locales, en vue de les recoller pour obtenir des équivalences stables au niveau «global». C'est dans cette perspective que nous avons consacré un chapitre de cette thèse aux équivalences de Morita entre algèbres fortement graduées.

La littérature sur le sujet des algèbres fortement graduées est extrêmement riche, avec notamment plusieurs articles de Dade, de Cohen et Montgomery, de Marcus, que nous citons en bibliographie. Les points de vue sur ces objets sont multiples. Dans notre deuxième chapitre, nous développons l'approche qui consiste à voir une algèbre fortement G-graduée $R = \bigoplus_{g\in G} R_g$ comme une algèbre $A = R_1$, munie d'une action faible du groupe G sur la catégorie ${}_A\mathbf{Mod}$ des A-modules. Cette approche repose sur des outils issus de la théorie des 2-catégories. Elle nous permet d'expliciter complètement les isomorphismes naturels qui apparaissent en relation avec le théorème de Morita, et qui se ramènent tous, *in fine*, au fameux «isomorphisme cher à Cartan» (et à Broué).

Nous utilisons cette approche fonctorielle des algèbres fortement graduées pour retrouver divers résultats plus ou moins classiques. Ce chapitre ne contient donc pas de résultat nouveau ; il vaut d'abord par son point de vue, qui ne se trouve pas couramment dans la littérature.

Cette approche fonctorielle nous permet surtout d'obtenir une nouvelle démonstration du résultat de Külshammer et Robinson évoqué plus haut. Avec les notations ci-dessus, nous utilisons le foncteur de Brauer Br_P pour construire un $(kG\bar{e}, kC_G(P)\bar{e}_P)$-bimodule M qui induit une équivalence de Morita entre les algèbres de bloc $kG\bar{e}$ et $kC_G(P)\bar{e}_P$, où $\bar{e}$ est un bloc de l'algèbre kG dont un groupe de défaut contient P, et $\bar{e}_P = \mathrm{br}_P(\bar{e})$ est le bloc correspondant de l'algèbre $kC_G(P)$. Nous montrons essentiellement que le bimodule M, si on lui impose d'induire une équivalence de Morita qui conserve la projectivité relative, est unique. Cette unicité très forte n'est pas explicitée dans le texte (disons que nous prouvons que notre module M est aussi «naturel» que possible), mais c'est bien elle qui va permettre ensuite le recollement.

Modules et catégories compatibles à la localisation

Après avoir bien compris les équivalences de Morita locales, il nous faudra construire un bimodule indécomposable au niveau «global», qui aura vocation à induire une équivalence stable entre deux algèbres de blocs, et dont la source sera un module d'endopermutation. Ce bimodule devra aussi induire les bimodules locaux que nous avons évoqués dans le paragraphe précédent. Il lui faut donc être compatible avec un outil de localisation, qui ne peut pas être simplement le foncteur de Brauer, puisque celui-ci ne fonctionne bien que pour les modules de p-permutation, c'est-à-dire les modules de source triviale.

Nous consacrons un chapitre à la construction de cet outil de localisation, à la définition des catégories de modules sur lesquelles il trouve à s'appliquer, et à l'explicitation de la correspondance de Puig pour ces modules. Comme base de ces constructions, nous reprenons la notion de module d'endopermutation «fusion-stable», développée par Linckelmann dans le langage des algèbres sources et des systèmes de fusion. Mais nous choisissons de nous placer systématiquement au niveau des algèbres de blocs, ce qui implique de travailler avec la catégorie de Brauer que nous avons définie plus haut.

Considérons un groupe fini G et un bloc e de l'algèbre $\mathcal{O}G$. Soit (P, e_P) une e-sous-paire du groupe G, et V un $\mathcal{O}P$-module d'endopermutation. Nous disons que V est fusion-stable relativement à la sous-paire (P, e_P) si, pour toute e-sous-paire (Q, e_Q) et pour tout couple de flèches $\phi, \psi : (Q, e_Q) \to (P, e_P)$ dans la catégorie de Brauer $\mathbf{Br}(G, e)$, les $\mathcal{O}Q$-modules d'endopermutation $\mathrm{Res}_\phi V$ et $\mathrm{Res}_\psi V$ sont compatibles. Si M est un $\mathcal{O}Ge$-module indécomposable, il existe une notion de e-sous-paire vortex de M, et de source associée. Nous disons que le $\mathcal{O}Ge$-module M est «Brauer-compatible» s'il admet une sous-paire vortex (P, e_P), associée à une source V qui est un $\mathcal{O}P$-module d'endopermutation fusion-stable relativement à la sous-paire (P, e_P). Nous généralisons ainsi la notion de $\mathcal{O}G$-module d'endo-p-permutation.

Nous montrons ensuite que la construction slash, évoquée plus haut pour les modules d'endopermutation sur des p-groupes, se généralise aux modules Brauer-compatible et aux sous-paires. Si M est un $\mathcal{O}Ge$-module Brauer-compatible et (Q, e_Q) une e-sous-paire du groupe G, il existe un $kN_G(Q, e_Q)$-module $M(Q, e_Q)$, pas tout à fait unique mais presque, tel que

$$\mathrm{Br}_Q(\mathrm{End}_{\mathcal{O}}(e_Q M)) \simeq \mathrm{End}_k(M(Q, e_Q)).$$

Cette construction possède plusieurs propriétés intéressantes du foncteur de Brauer, notamment la transitivité. Il lui manque toutefois l'unicité, mais aussi, *a priori*, la fonctorialité.

Pour obtenir cette fonctorialité, il nous faut d'abord définir une catégorie de module sur laquelle celle-ci puisse s'appliquer. En effet, nous avons déjà dit que la catégorie des $\mathcal{O}G$-modules d'endo-p-permutation n'est pas stable par sommes directes. Le même phénomène se produit pour les modules Brauer-compatibles. Pour le maîtriser, nous définissons la compatibilité de deux modules d'endopermutation fusion-stables relativement à des e-sous-paires fixées du groupe G. Nous disons qu'une sous-catégorie ${}_{\mathcal{O}Ge}\mathbf{M}$ de la catégorie ${}_{\mathcal{O}Ge}\mathbf{Mod}$ est «Brauer-compatible» si elle est stable par sommes directes, et si les sources de deux facteurs directs indécomposables d'objets de ${}_{\mathcal{O}Ge}\mathbf{M}$ sont toujours des modules d'endopermutation fusion-stables et compatibles relativement aux sous-paires vortex concernées.

Un exemple de catégorie Brauer-compatible est la catégorie ${}_{\mathcal{O}Ge}\mathbf{Perm}$ des $\mathcal{O}Ge$-modules de p-permutation, qui sont les modules de sources triviales. Mais il en existe bien d'autres, associées par exemple à chacun des modules d'endopermutation non triviaux qui sont fusion-stable relativement à une e-sous-paire maximale (D, e_D). Dès lors qu'une catégorie Brauer-compatible ${}_{\mathcal{O}Ge}\mathbf{M}$ est fixée, nous montrons que la construction slash décrite ci-dessus permet de construire, pour toute e-sous-paire (Q, e_Q) du groupe G, un «foncteur slash»

$$\mathrm{Sl}_{(Q,e_Q)} : {}_{\mathcal{O}Ge}\mathbf{M} \to {}_{kN_G(Q,e_Q)\bar{e}_Q}\mathbf{Mod},$$

qui induit, pour tout couple d'objets L, M de la catégorie ${}_{\mathcal{O}Ge}\mathbf{M}$, un isomorphisme

$$\mathrm{Br}_Q(\mathrm{Hom}_{\mathcal{O}}(e_Q L, e_Q M)) \; \simeq \; \mathrm{Hom}_k(\mathrm{Sl}_{(Q,e_Q)}(L), \mathrm{Sl}_{(Q,e_Q)}(M)).$$

Ce foncteur n'est pas tout à fait unique ; il peut être tordu par un caractère linéaire du groupe $N_G(Q, e_Q)/QC_G(Q)$. À cette réserve près, il se comporte essentiellement comme le foncteur de Brauer pour les modules de p-permutation.

En particulier, nous pouvons utiliser un foncteur slash $\mathrm{Sl}_{(P,e_P)}$ en vue d'expliciter la correspondance de Puig pour les modules Brauer-compatibles. Si M est un objet indécomposable de la catégorie Brauer-compatible ${}_{\mathcal{O}Ge}\mathbf{M}$, il est caractérisé par le quadruplet $(P, e_P, V, M(P, e_P))$, où (P, e_P) est une sous-paire vortex de M, V est la source de M relativement à cette sous-paire, et $M(P, e_P) = \mathrm{Sl}_{(P,e_P)}(M)$ est un $kN_G(P, e_P)/P$-module projectif indécomposable. Il faut toutefois noter que la correspondance que nous venons de décrire dépend du choix d'un foncteur slash $\mathrm{Sl}_{(P,e_P)}$ qui, nous l'avons mentionné, n'est pas tout à fait unique.

Contrôle de la fusion forte et équivalences stables

Au seuil de ce quatrième et dernier chapitre, nous avons tous les éléments en main pour démontrer notre résultat principal, qui est le suivant.

Théorème. *On suppose ici p impair. Soit G un groupe fini, et $H = C_G(P)$ le centralisateur d'un p-sous-groupe P de G. Soit e un bloc de l'algèbre $\mathcal{O}G$, et e_P un bloc de l'algèbre $\mathcal{O}H$ tel que $\bar{e}_P \,\mathrm{br}_P(e) \neq 0$. Supposons qu'il existe une e-sous-paire maximale (D, e_D) du groupe G telle que*

- *$(P, e_P) \leqslant (D, e_D)$ et $D \leqslant H$;*
- *pour toute e-sous-paire non triviale (Q, e_Q) de (D, e_D), on a la factorisation*

$$N_G(Q, e_Q) = O_{p'}(C_G(Q))\, C_{N_G(Q,e_Q)}(P) \ ;$$

- *le groupe de défaut D est abélien, ou son centre n'est pas cyclique.*

Alors il existe une équivalence stable de type Morita entre les algèbres de bloc $\mathcal{O}Ge$ et $\mathcal{O}C_G(P)e_P$.

On notera que cet énoncé décrit la situation d'un contre-exemple minimal au Z_p^*-théorème, tout en la généralisant quelque peu, puisque les p-sous-groupes sont remplacés par des e-sous-paires. En fait, on suppose que le centralisateur $C_G(P)$ contrôle la e-fusion (en un sens renforcé) dans le groupe G, et pas nécessairement la p-fusion. On notera aussi une hypothèse technique, portant sur le groupe de défaut D, dont nous expliquerons plus loin la justification.

Pour démontrer ce théorème, il nous faut construire un $(\mathcal{O}Ge, \mathcal{O}C_G(P)e_P)$-bimodule M, que nous considérerons comme un $\mathcal{O}(G \times C_G(P))(e \otimes e_P^{\circ})$-module indécomposable. Nous utilisons pour cela la correspondance de Puig, telle que nous l'avons explicitée au paragraphe précédent. Pour commencer, le module M admettra comme vortex la $(e \otimes e_P^{\circ})$-sous-paire $(\Delta D, e_D \otimes e_D^{\circ})$ du groupe $G \times C_G(P)$.

Il nous faut ensuite identifier la source de M relativement à cette sous-paire vortex. Pour chaque sous-paire non triviale (Q, e_Q) de la e-sous-paire maximale (D, e_D), nous disposons d'une équivalence de Morita locale entre les algèbres de blocs $kN_G(Q, e_Q)\bar{e}_Q$ et $kC_{N_G(Q,e_Q)}(P)\,\mathrm{br}_P(\bar{e}_Q)$. Cette équivalence a été brièvement étudiée au premier chapitre. Elle est induite par un bimodule $M_{(Q,e_Q)}$, qui est obtenu grâce à la construction slash. Nous revenons en détail sur cette construction, et caractérisons très précisément la source du bimodule $M_{(Q,e_Q)}$. En particulier, nous lui associons un $kN_D(Q)$-module V_Q, qui est fusion-stable relativement à la e-sous-paire $(N_D(Q), e_{N_D(Q)})$.

Nous obtenons alors une famille $(V_Q)_{1 \neq Q \leqslant D}$ de modules d'endopermutation, indexée par les sous-groupes non triviaux du groupe de défaut D. Grâce à la façon canonique dont elle a été obtenue, cette famille vérifie des conditions de compatibilité très précises, au regard de la conjugaison et de la construction slash. Nous utilisons alors des résultats sur le recollement des modules d'endopermutation, dus à Puig dans le cas où D est abélien, à Bouc et Thévenaz dans le cas où le centre $Z(D)$ n'est pas cyclique. Nous en déduisons l'existence d'un $\mathcal{O}D$-module d'endopermutation V tel que, pour tout sous-groupe non trivial Q de D, on ait

$$\mathrm{Br}_Q(\mathrm{End}_{\mathcal{O}}(V)) \simeq \mathrm{End}_k(V_Q).$$

De plus, ce $\mathcal{O}D$-module d'endopermutation est fusion-stable relativement à la e-sous-paire maximale (D, e_D). Après l'avoir ramené sur le p-groupe diagonal ΔD, nous ferons de V la source du module M relativement à la sous-paire vortex $(\Delta D, e_D \otimes e_D^{\circ})$.

Il nous reste à déterminer le troisième invariant associé au module M. Nous connaissons son vortex et sa source, nous pouvons donc situer M dans une catégorie Brauer-compatible ${}_{\mathcal{O}(G\times C_G(P))(e\otimes e_P^{\circ})}\mathbf{M}$ bien définie. Nous savons de plus que le module slash $V(P)$ associé à la source V est le kD-module trivial k. Ceci implique l'existence d'un unique foncteur slash

$$\mathrm{Sl}_{(\Delta P, e_P \otimes e_P^{\circ})} \ : \ {}_{\mathcal{O}(G\times C_G(P))e\otimes e_P^{\circ}}\mathbf{M} \ \to \ {}_{k(C_G(P)\times C_G(P))\bar{e}_P\otimes \bar{e}_P^{\circ}}\mathbf{Perm}$$

Ceci nous permet d'utiliser une version légèrement amendée de la correspondance de Puig, et de caractériser M comme l'unique $(\mathcal{O}Ge, \mathcal{O}C_G(P)e_P)$-bimodule indécomposable, de sous-paire vortex $(\Delta D, e_D \otimes e_D^\circ)$ et de source V, tel que

$$\mathrm{Sl}_{(\Delta P, e_P \otimes e_P^\circ)}(M) \;\simeq\; kC_G(P)\bar{e}_P.$$

Si (Q, e_Q) est une sous-paire non triviale du groupe D, nous montrons ensuite que la construction slash, appliquée au bimodule M relativement à la sous-paire $(\Delta Q, e_Q \otimes e_{PQ})$, permet de retrouver le bimodule $M_{(Q,e_Q)}$ qui induit une équivalence de Morita $kN_G(Q, e_Q)\bar{e}_Q \sim kC_{N_G(Q,e_Q)}\bar{e}_{PQ}$.

Pour conclure, nous utilisons un théorème de Linckelmann établissant que le bimodule M, puisqu'il induit des équivalences de Morita locales dans les conditions que nous venons brièvement de décrire, induit nécessairement une équivalence stable entre les algèbres de blocs $\mathcal{O}Ge$ et $\mathcal{O}C_G(P)e_P$.

Summary of results

In this summary, we fix a p-modular system $(\mathbb{K}, \mathcal{O}, k)$, and we feel free to use the notations and definitions that will be set up in the subsequent chapters of this thesis. In Chapter 1, we review the preexisting material that we use in the rest of the thesis. The content of that chapter is scarcely original, except maybe the definition of the Brauer functor attached to a subpair of a finite group. Here is a brief account of the main definitions and theorems of the other three chapters.

Chapter 2. Morita equivalences and strongly graded rings

In Section 2.1, we give a thorough description of Morita's theory of bimodules and functors between module categories, in terms of adjoint functors between 2-categories. In the rest of Chapter 2, we use that description as a dictionary, in order to translate the theory of strongly group-graded algebras into a functorial language. The central result, which appears in Section 2.2, could be stated a follows.

Theorem. *Let G be a group. Let $R = \bigoplus_{g\in G} R_g$ be a strongly G-graded algebra, with unit component R_1. Then the family of functors $(R_g \otimes_{R_1} -)_{g\in G}$ induces an action $\star$ of the group G on the category ${}_{R_1}\mathbf{Mod}$ of R_1-modules. Conversely, let A be an $\mathcal{O}$-algebra and $\star$ be an action of the group G on the category ${}_A\mathbf{Mod}$.*

Then the $\mathcal{O}$-module $A_\star G = \bigoplus_{g\in G} g_\star A$ admits a natural structure of strongly G-graded algebra, with a unit component $1_\star A$ that is isomorphic to the algebra A.

These constructions induce an equivalence between the category $\mathbf{SGAlg}(G,\mathcal{O})$ of strongly G-graded $\mathcal{O}$-algebras, and the category $\mathbf{EqAlg}(G,\mathcal{O})$ of "G-equivariant algebras" over the ring $\mathcal{O}$, i.e.*, $\mathcal{O}$-algebras endowed with an action of the group G on their categories of modules.*

This theorem suggests a functorial approach, which we use to review the classical theory of strongly group-graded algebras, and some of its applications to the modular representation theory.

Chapter 3. Brauer-friendly modules and categories

This chapter aims at defining a class of modules over block algebras and a construction that generalize the properties of the Brauer functor with respect to the class of p-permutation modules.

In the first two sections, we define the vertex subpairs of an indecomposable module over a block algebra, following [Si-1990]. We adapt from [Li-2013] the definition of a fusion-stable endopermutation source module with respect to a given subpair, and we complete it with a relation of compatibility between such source modules. For a given block algebra $\mathcal{O}Ge$, we call an $\mathcal{O}Ge$-module Brauer-friendly if it is a direct sum of indecomposable modules with compatible fusion-stable endopermutation sources. If A is a source algebra of the block algebra $\mathcal{O}Ge$ with respect to a defect group D and a source idempotent i, an $\mathcal{O}Ge$-module M is Brauer-friendly if, and only if, the corresponding A-module iM is an endopermutation $\mathcal{O}D$-module.

We call a category of $\mathcal{O}Ge$-modules Brauer-friendly if the direct sum of any two objects of that category is a Brauer-friendly module (*i.e.*, the two objects are Brauer-friendly and compatible). We give the following generalisation of both the Brauer functor and the slash construction, which is proven in Sections 3.3 and 3.4.

Theorem. *Let G be a finite group, and e be a block of the group algebra $\mathcal{O}G$. Let ${}_{\mathcal{O}Ge}\mathbf{M}$ be a Brauer-friendly category of $\mathcal{O}Ge$-modules. For any e-subpair (P,e_P) of the group G, there exists a Brauer-friendly category ${}_{kN_G(P,e_P)\bar{e}_P}\mathbf{M}$ of $kN_G(P,e_P)\bar{e}_P$-modules on which P acts trivially, and there exists a "(P,e_P)-slash* *functor"*

$$\mathrm{Sl}_{(P,e_P)} : {}_{\mathcal{O}Ge}\mathbf{M} \to {}_{kN_G(P,e_P)\bar{e}_P}\mathbf{M},$$

with the following features. Besides being a functor in the usual sense, $\mathrm{Sl}_{(P,e_P)}$ *maps any morphism of* $\mathcal{O}P$*-modules between two objects* L *and* M *of the category* ${}_{\mathcal{O}Ge}\mathbf{M}$ *to a* k*-linear map between the corresponding objects* $\mathrm{Sl}_{(P,e_P)}(L)$ *and* $\mathrm{Sl}_{(P,e_P)}(M)$ *of the category* ${}_{kN_G(P,e_P)\bar{e}_P}\mathbf{M}$*. This mapping is functorial and induces an isomorphism of* $k(C_G(P) \times C_G(P))\Delta N_G(P, e_P)$*-modules*

$$\mathrm{Br}_{(\Delta P, e_P \otimes e_P^{\circ})}(\mathrm{Hom}_{\mathcal{O}}(L, M)) \;\simeq\; \mathrm{Hom}_k(\mathrm{Sl}_{(P,e_P)}(L), \mathrm{Sl}_{(P,e_P)}(M)).$$

The (P, e_P)*-slash functor* $\mathrm{Sl}_{(P,e_P)}$ *is unique up to twisting by a linear character of the group* $N_G(P, e_P)/PC_G(P)$ *with values in* $k^{\times}$*.*

Despite their non-unicity, the behaviour of slash functors over Brauer-friendly categories is quite similar to that of the Brauer functor over categories of p-permutation, especially with respect to composition and conjugacy. The most important application of these functors is the following parametrisation of indecomposable modules, which is proven in Section 3.5 and may be seen as an instance of the Puig correspondence defined in [Pu-1988a].

Theorem. *Let* G *be a finite group,* e *be a block of the algebra* $\mathcal{O}G$*, and* (P, e_P) *be an* e*-subpair of the group* G*. Let* V *be an indecomposable endopermutation* $\mathcal{O}P$*-module that is fusion-stable in the group* G *with respect to the subpair* (P, e_P)*. Let* ${}_{\mathcal{O}Ge}\mathbf{M}$ *be a Brauer-friendly subcategory of* ${}_{\mathcal{O}Ge}\mathbf{Mod}$ *such that every indecomposable* $\mathcal{O}Ge$*-module with source triple* (P, e_P, V) *is an object of* ${}_{\mathcal{O}Ge}\mathbf{M}$*. Let* $\mathrm{Sl}_{(P,e_P)} : {}_{kGe}\mathbf{M} \to {}_{kN_G(P,e_P)\bar{e}_P}\mathbf{Mod}$ *be a* (P, e_P)*-slash functor. Then the mapping* $M \mapsto \mathrm{Sl}_{(P,e_P)}(M)$ *induces a one-to-one correspondence between the isomorphism classes of indecomposable* $\mathcal{O}Ge$*-modules with source triple* (P, e_P, V) *and the isomorphism classes of indecomposable* $kN_G(P, e_P)\bar{e}_P$*-modules with vertex* P *and trivial source,* i.e.*, the isomorphism classes of projective indecomposable* $k(N_G(P, e_P)/P)\bar{e}_P$*-modules.*

In this theorem, the choice of the Brauer-friendly category ${}_{\mathcal{O}Ge}\mathbf{M}$ is unimportant once the source triple (P, e_P, V) has been fixed. Thus the indecomposable $\mathcal{O}Ge$-module M is characterised by the triple $((P, e_P), V, \mathrm{Sl}_{(P,e_P)}(M))$. Unfortunately, the slash functor $\mathrm{Sl}_{(P,e_P)}$ is not uniquely defined, so this parametrisation of indecomposable Brauer-friendly modules needs to be handled with care.

Chapter 4. Strong fusion control and stable equivalences

The Z_p^*-theorem may be seen as a statement that connects an assumption about p-fusion control to a conclusion about a Morita equivalence between principal blocks of finite groups. For an odd prime p, there is no modular proof of the Z_p^*-theorem, but there is a modular proof of a weaker statement, which leads to a stable equivalence between principal blocks. In chapter 4, we generalise that result of Broué to nonprincipal blocks.

The first section reviews classical results that are related to the Z_p^*-theorem. The second section is devoted to a new proof of the main result of [KR-1986], *i.e.*, a Morita equivalence between block algebras that arises from a factorisation of the form $G = O_{p'}(G)\, C_G(P)$, where P is a p-subgroup of the finite group G. We obtain a great deal of new information about the bimodule that induces that Morita equivalence.

The third section contains the proof of the following theorem, which may be suggestive of a generalisation of the Z_p^*-theorem to nonprincipal blocks (but this generalisation, or even a precise conjecture, is out of the bounds of the present thesis).

Theorem. *We assume that the prime p is odd. Let G be a group, and e be a block of the algebra $\mathcal{O}G$. Let $H = C_G(P)$ be the centraliser of a p-subgroup P of G, and e_P be a block of the algebra $\mathcal{O}H$ such that $\bar{e}_P \operatorname{br}_P(e) \neq 0$. Suppose that the group G admits a maximal e-subpair (D, e_D) such that*

- $(P, e_P) \leqslant (D, e_D)$ *and* $D \leqslant H$*;*
- $N_G(Q, e_Q) = O_{p'}(C_G(Q))\, N_H(Q, e_Q)$
  for any non-trivial e-subpair $(Q, e_Q) \leqslant (D, e_D)$.

Suppose moreover that the defect group D is abelian, or that its center $Z(D)$ is not cyclic. Then there exists a stable equivalence of Morita type between the block algebras $\mathcal{O}Ge$ and $\mathcal{O}He_P$.

Chapitre 1
Préliminaires

Chapter 1
Preliminaries

This chapter gathers the general definitions and notations that we will use in the remaining of the thesis. Most of those are classical. We do not prove the results that we quote here, but we try to give precise references to the literature, where the reader can find more details and proofs.

Let p be a prime number. A p-modular system is a triple $(\mathbb{K}, \mathcal{O}, k)$, where $\mathcal{O}$ is a complete discrete valuation ring, $\mathbb{K}$ is its fraction field, and k is its residue field of characteristic p. To avoid any rationality issue, we always assume that the fraction field $\mathbb{K}$ is «big enough». In general, this fraction field should be of characteristic zero, but we also allow the case $\mathbb{K} = \mathcal{O} = k$, so that every result that is proven over the ring $\mathcal{O}$ remains true over the field k. These notations will hold throughout the thesis. Notice, however, that most of the constructions and results of chapter 2 still hold when $\mathcal{O}$ is replaced by an arbitrary commutative unitary ring.

A finite group G is a p-group if its order is a power of the prime p; it is a p'-group if its order is not a multiple of the prime p. We denote by $O_{p'}(G)$ the largest normal p'-subgroup of G, which is a characteristic subgroup of G. The group G is said to be p-nilpotent if it is the semidirect product of a p-subgroup by a normal p'-subgroup, *i.e.*, if it admits the factorisation $G = O_{p'}(G)P$, where P is a Sylow p-subgroup of G. For any group G, we denote by $\mathcal{O}G$ the group algebra of the group G over the ring $\mathcal{O}$, *i.e.*, the free $\mathcal{O}$-module over the set G, endowed with the multiplication law that bilinearly extends the group law of G.

An $\mathcal{O}$-algebra will always mean a unitary $\mathcal{O}$-algebra (and a ring means a unitary ring). The unity of an $\mathcal{O}$-algebra A will be denoted by 1_A, while the boldface $\mathbf{1}_X$ will stand for the identity arrow of an object X in some category. An algebra morphism $A \to B$ is an $\mathcal{O}$-linear multiplicative map that sends 1_A to 1_B. We denote by A^{op} the opposite algebra of an $\mathcal{O}$-algebra A. For any integer d, we denote by $M_d(\mathcal{O})$ the $\mathcal{O}$-algebra of matrices of size $d \times d$; this is the zero $\mathcal{O}$-algebra if $d = 0$. An $\mathcal{O}$-algebra A is said to be a matrix algebra if it is isomorphic to $M_d(\mathcal{O})$ for some integer d.

Let A be an $\mathcal{O}$-algebra. An idempotent of A is a non-zero element $i \in A$ such that $i^2 = i$. The idempotent i is said to be decomposable in A if there exist idempotents i_1 and i_2 of A such that $i_1 i_2 = i_2 i_1 = 0$ and $i = i_1 + i_2$. The idempotent i is said to be primitive in A if it is not decomposable. A block of the algebra A is a primitive idempotent of the center $Z(A)$. If e is a block of A, the corresponding block algebra is the algebra $eA = Ae$.

If M and N are $\mathcal{O}$-modules, the notation $M \otimes N$ will always stand for the tensor product of M and N over the ring $\mathcal{O}$. In particular, if A and B are two $\mathcal{O}$-algebras, an (A, B)-bimodule will mean an $A \otimes_{\mathcal{O}} B^{\mathrm{op}}$-module, where B

is considered as acting on the right.

If G is a group, we denote by $\Delta G = \{(g,g); g \in G\}$ the diagonal subgroup of the direct product $G \times G$. For an element g of the group G and an object X, the notation ${}^gX$ stands for the object gXg^{-1} whenever this makes sense. For any element $x = \sum_{g\in G} x_g g \in \mathcal{O}G$, we denote by x^{o} the element $\sum_{g\in G} x_g g^{-1} \in \mathcal{O}G$. For any two groups G and H, we identify the group algebra $\mathcal{O}(G \times H)$ with the tensor product $\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}H$ in the obvious way. As a consequence, an $(\mathcal{O}G, \mathcal{O}H)$-bimodule M may be considered as an $\mathcal{O}(G \times H)$-bimodule by setting

$$(x,y)\cdot m = x \cdot m \cdot y^{\mathrm{o}} \qquad \text{for any } x \in \mathcal{O}G,\ y \in \mathcal{O}H \text{ and } m \in M.$$

Whenever x is an element defined over the ring $\mathcal{O}$ (*e.g.*, an element $x \in \mathcal{O}G$), we denote by $\bar{x}$ its reduction to an element defined over the field k, via the natural projection map $\mathcal{O} \to k$ (*e.g.*, $\bar{x} \in kG$). For instance, it is well-known that the projection map $\mathcal{O}G \mapsto kG$ induces a one-to-one correspondence between the blocks of the algebra $\mathcal{O}G$ and the blocks of the algebra kG.

1.1 Morita equivalences and stable equivalences

Let A be an $\mathcal{O}$-algebra that is a finitely generated free $\mathcal{O}$-module. We denote by ${}_A\mathbf{Mod}$ the abelian category of finitely generated left A-modules and morphisms of left A-modules. From now on, an A-module will always be assumed to be on the left and finitely generated. An A-module P is said to be projective if it is a projective object of the category ${}_A\mathbf{Mod}$, *i.e.*, if it is isomorphic to a direct summand of a free A-module. Let X and Y be two A-modules. A morphism of A-modules $u : X \to Y$ is said to be projective if there exists a projective A-module P and a commutative diagram

$$\overbrace{X \longrightarrow P \longrightarrow Y}^{u}\ .$$

We denote by $\mathrm{Hom}_A^{pr}(X,Y)$ the $\mathcal{O}$-submodule of $\mathrm{Hom}_A(X,Y)$ consisting of the projective morphisms. A stable morphism of A-modules $\tilde{u} : X \to Y$ is an element of the quotient $\mathcal{O}$-module

$$\mathrm{Hom}_A^{st}(X, Y) = \mathrm{Hom}_A(X, Y)/\,\mathrm{Hom}_A^{pr}(X, Y).$$

The stable morphisms can be composed consistently. We denote by $\mathbf{Stab}({}_A\mathbf{Mod})$ the stable category of A-modules, *i.e.*, the triangulated category of A-modules and stable morphisms of A-modules.

Let B be a second $\mathcal{O}$-algebra. An (A, B)-bimodule M is said to induce a Morita equivalence between the algebras A and B if, and only if, there exists a (B, A)-bimodule N and isomorphisms

$$\begin{aligned} M \otimes_B N &\simeq A \qquad \text{as } (A, A)\text{-bimodules;} \\ N \otimes_A M &\simeq B \qquad \text{as } (B, B)\text{-bimodules.} \end{aligned}$$

This is equivalent to requiring that the $\mathcal{O}$-linear functor $M \otimes_B - : {}_B\mathbf{Mod} \to {}_A\mathbf{Mod}$ be an equivalence of categories. A classical theorem of Morita states that any $\mathcal{O}$-linear equivalence of categories ${}_B\mathbf{Mod} \to {}_A\mathbf{Mod}$ is isomorphic to a functor of the type $M \otimes_B -$ for some (A, B)-bimodule M.

An (A, B)-bimodule M is said to induce a stable equivalence of Morita type between the algebras A and B if, and only if, there exists a (B, A)-bimodule N, projective (A, A)-bimodules U and U', projective (B, B)-bimodules V and V', and isomorphisms

$$\begin{aligned} (M \otimes_B N) \oplus U &\simeq A \oplus U' \qquad \text{as } (A, A)\text{-bimodules;} \\ (N \otimes_A M) \oplus V &\simeq B \oplus V' \qquad \text{as } (B, B)\text{-bimodules.} \end{aligned}$$

This is equivalent to requiring that the $\mathcal{O}$-linear triangulated functor $M \otimes_B - :$ $\mathbf{Stab}({}_B\mathbf{Mod}) \to \mathbf{Stab}({}_A\mathbf{Mod})$ be an equivalence of categories. We will not be concerned with equivalences between stable categories of modules that are not of Morita type. More details about stable equivalences of Morita type can be found in [Rq-2001].

1.2 Interior algebras

Let G be a group. A G-algebra over the ring $\mathcal{O}$ is a pair (A, γ), where A is an algebra over the ring $\mathcal{O}$ and $\gamma : G \to \mathrm{Aut}_{\mathcal{O}\text{-}\mathbf{Alg}}(A)$ is a group morphism.

Let S be a normal subgroup of G. An S-interior G-algebra over the ring $\mathcal{O}$, as defined in [Pu-2002, 4.2], is a triple (A, ι, γ) where A is an $\mathcal{O}$-algebra and $\iota : S \to A^\times$, $\gamma : G \to \mathrm{Aut}_{\mathcal{O}\mathbf{Alg}}(A)$ are two group morphisms such that

$$\gamma(s)(a) = \iota(s)a\,\iota(s)^{-1} \quad \text{and} \quad \iota(gsg^{-1}) = \gamma(g)(\iota(s))$$

for any $a \in A$, $s \in S$ and $g \in G$. When $S = G$, we say that A is a G-interior algebra, or an interior G-algebra. We consider the action of the group G on the $\mathcal{O}$-module A as a diagonal action. Together with the left and right multiplication by elements of the form $\iota(s)$ for $s \in S$, this makes A an $\mathcal{O}(S\times S)\Delta G$-module. Let A and B be two S-interior G-algebras. A map $u : A \to B$ is a morphism of S-interior G-algebras if it is a multiplicative map and a morphism of $\mathcal{O}(S \times S)\Delta G$-modules. We may not require it to send the unity 1_A to the unity 1_B. The map u is said to be an embedding if it induces an isomorphism $A \simeq u(1_A)\,B\,u(1_A)$.

As an example, the group algebra $\mathcal{O}S$ of the normal subgroup S is naturally an S-interior G-algebra. More generally, if $e \in \mathcal{O}S$ is a G-invariant idempotent, then $\mathcal{O}Se$ is again an S-interior G-algebra, where the group G acts on the algebra $\mathcal{O}S$ by conjugation, and the group S is sent to the group of units $(\mathcal{O}Se)^\times$ by the map $s \mapsto se$. This example is a fundamental one. Indeed, for any S-interior G-algebra A over the ring $\mathcal{O}$, the S-interior structure map $\iota : S \to A^\times$ can be linearly extended to a unique map $\tilde{\iota} : \mathcal{O}S \to A$, which is a morphism of S-interior G-algebras.

Puig's induction of interior algebras can be readily generalised to partially interior algebras. Let G be a group, and S be a normal subgroup of G; let G' be a group that contains G, and S' be a normal subgroup of G' that contains S; let A be an S-interior G-algebra over the ring $\mathcal{O}$. Then A is an $\mathcal{O}(S\times S^{\mathrm{op}})\Delta G$-module. We consider the induced module

$$A' \;=\; \mathrm{Ind}_{(S\times S)\Delta G}^{(S'\times S')\Delta G'} A \;=\; \mathcal{O}(S' \times S')\Delta G' \otimes_{\mathcal{O}(S\times S)\Delta G} A.$$

The $\mathcal{O}$-module A' is generated by elements of the form $(g, h) \otimes a$, for $(g, h) \in$

$(S' \times S')\Delta G'$ and $a \in A$. To stick to Puig's notations in the case of purely interior algebras, we denote the element $(g,h)\otimes a$ by $g \otimes a \otimes h^{-1}$. We define an $\mathcal{O}$-bilinear multiplication law on A' by setting, for any two elements (g,h) and (g',h') of the group $(S' \times S')\Delta G'$ and any two elements a and a' of the algebra A'.

$$(g\otimes a\otimes h^{-1})\cdot(g'\otimes a'\otimes h'^{-1}) \;=\; \left|\begin{array}{ll} g\otimes a\gamma(h^{-1}g')(a')\otimes h^{-1}g'h'^{-1} & \text{if } h^{-1}g' \in G; \\ 0 & \text{if not.} \end{array}\right.$$

This makes A' a unitary algebra, with $1_{A'} = \sum_{g\in G'/G} g\otimes 1_A \otimes g^{-1}$. Furthermore, the action of the group $(S' \times S')\Delta G'$ defines a unique structure of S'-interior G'-algebra on A'. If $S = G$ and $S' = G'$, the induced algebra $\mathrm{Ind}_{G\times G}^{G'\times G'} A$ is just denoted by $\mathrm{Ind}_G^{G'} A$. The context usually makes the notation unambiguous.

There is one important example that we will use in Chapter 4. Let G be a finite group, let $S \leqslant T$ be two normal subgroups of G, and let b be a block of the group algebra $\mathcal{O}S$. We denote by G_b the stabiliser of the block b in the group G, and by b' the sum of the distinct G-conjugates of the block b. The block algebra $\mathcal{O}Sb$ has a natural structure of S-interior G_b-algebra, and there is a natural isomorphism of T-interior G-algebras

$$\tilde{\iota} : \mathcal{O}Tb' \;\to\; \mathrm{Ind}_{(S\times S)\Delta G_b}^{(T\times T)\Delta G} \mathcal{O}Sb.$$

As the notation suggests, the map $\tilde{\iota}$ is induced by the interior structure map $\iota : T \to \left(\mathrm{Ind}_{(S\times S)\Delta G_b}^{(T\times T)\Delta G} \mathcal{O}Sb\right)^{\times}$ of the T-interior G-algebra $\mathrm{Ind}_{(S\times S)\Delta G_b}^{(T\times T)\Delta G} \mathcal{O}Sb$.

1.3 Permutation and endopermutation modules

Let G be a finite group. Let X be a finite G-set, *i.e.*, a finite set endowed with an action of the group G. We denote by $\mathcal{O}X$ the free $\mathcal{O}$-module on the set X. The action of the group G on the set X has a unique extension to a linear action of G on the $\mathcal{O}$-module $\mathcal{O}X$, which makes $\mathcal{O}X$ an $\mathcal{O}G$-module.

A permutation $\mathcal{O}G$-module is an $\mathcal{O}G$-module that is isomorphic to $\mathcal{O}X$ for some finite G-set X. A p-permutation $\mathcal{O}G$-module is an $\mathcal{O}G$-module that is isomorphic to a direct summand of a permutation $\mathcal{O}G$-module. We denote by ${}_{\mathcal{O}G}\mathbf{Perm}$ the category of p-permutation $\mathcal{O}G$-modules and morphisms of $\mathcal{O}G$-

module. This subcategory of ${}_{\mathcal{O}G}\mathbf{Mod}$ is closed under finite direct sums and direct summands. If H is a subgroup of H, then the restriction $\operatorname{Res}^G_H M$ of a p-permutation $\mathcal{O}G$-module M is again a p-permutation $\mathcal{O}H$-module, and the induction $\operatorname{Ind}^G_H L$ of a p-permutation $\mathcal{O}H$-module L is again a p-permutation $\mathcal{O}G$-module.

Let L and M be $\mathcal{O}G$-modules. The $\mathcal{O}$-module $\operatorname{Hom}_{\mathcal{O}}(L, M)$ is endowed with a diagonal action of the group G, which is defined by setting

$$ {}^g u(l) = g \cdot u(g^{-1} \cdot l) \qquad \text{for any } g \in G,\, u \in \operatorname{Hom}_{\mathcal{O}}(L, M) \text{ and } l \in L. $$

This action makes $\operatorname{Hom}_{\mathcal{O}}(L, M)$ an $\mathcal{O}G$-module. For $L = M$, it makes the endomorphism algebra $\operatorname{End}_{\mathcal{O}}(M)$ an $\mathcal{O}G$-module, and a G-algebra over the ring $\mathcal{O}$. An endopermutation (*resp.* endo-p-permutation) $\mathcal{O}G$-module is an $\mathcal{O}G$-module M such that the endomorphism algebra $\operatorname{End}_{\mathcal{O}}(M)$ is a permutation (*resp.* p-permutation) $\mathcal{O}G$-module.

If P is a p-group and Q is a subgroup of P, then a classical theorem of Green [Gr-1959, Theorem 8] states that the induced $\mathcal{O}P$-module $\operatorname{Ind}^P_Q V$ is indecomposable whenever V is an indecomposable $\mathcal{O}Q$-module. This implies that any p-permutation (*resp.* endo-p-permutation) $\mathcal{O}P$-module is necessarily a permutation (*resp.* endopermutation) module. More details on these types of modules can be found in [Br-1985] for p-permutation modules, in [Da-1978] for endopermutation modules over p-groups, and in [Ur-2006] for endo-p-permutation modules over general finite groups.

Let G be a finite group (which might be a p-group). The category of endo-p-permutation $\mathcal{O}G$-modules is not very interesting, because it is not closed under finite direct sums. Two endo-p-permutation $\mathcal{O}G$-modules L and M are said to be compatible if their direct sum $L \oplus M$ is an endo-p-permutation $\mathcal{O}G$-module or, equivalently, if the diagonal action of the group G on $\operatorname{Hom}_{\mathcal{O}}(L, M)$ makes it a p-permutation $\mathcal{O}G$-module. If H is any subgroup of G, then the restriction $\operatorname{Res}^G_H M$ of an endo-p-permutation $\mathcal{O}G$-module M is again an endo-p-permutation $\mathcal{O}H$-module. However, this is not true for induction. The compatibility relation can be used to give a criterion for the induction $\operatorname{Ind}^G_H L$ of an endo-p-permutation $\mathcal{O}H$-module L to be an endo-p-permutation $\mathcal{O}G$-module. This is [Da-1978, Lemma 6.8] and [Ur-2006, Proposition 2.10].

Let P be a p-group. An $\mathcal{O}P$-module V is said to be capped if it admits an indecomposable direct summand with vertex P (see Section 1.7 for the definition of a vertex). If V is a capped endopermutation $\mathcal{O}P$-module, then the reduction $\bar{V} = k \otimes_{\mathcal{O}} V$ is a capped endopermutation kP-module. A P-algebra over the field k is called a Dade P-algebra if it is isomorphic to the endomorphism algebra $\mathrm{End}_k(\bar{V})$ of some capped endopermutation kP-module $\bar{V}$.

The compatibility relation is an equivalence relation on the set of capped endo-permutation kP-modules. We denote by $[\bar{V}]$ the equivalence class of a capped endopermutation kP-module $\bar{V}$. The set of equivalences classes is denoted by $\mathcal{D}(P)$. It is endowed with an operation denoted by $+$, and such that $[\bar{V}] + [\bar{W}] := [\bar{V} \otimes_k \bar{W}]$ for any two capped endopermutation kP-module $\bar{V}$ and $\bar{W}$. This operation makes $\mathcal{D}(P)$ an abelian group, which is known as the Dade group of the p-group P. This Dade group can also be seen as the set of compatibility classes of Dade P-algebras.

The capped endopermutation kP-modules have been classified by Dade for an abelian p-group P, and by Bouc, Carlsson and Thévenaz for an arbitrary p-group P. A survey can be found in [Th-2007]. In particular, for any p-group P, and any capped endopermutation kP-module $\bar{V}$, it follows from the classification that there exists a capped endopermutation $\mathcal{O}P$-module V sur that $\bar{V} \simeq k \otimes_{\mathcal{O}} V$.

For a general finite group G, a generalisation of the Dade group has been defined in [La-2012, Section 7.5]. An element of the Dade group $\mathcal{D}(G)$ is an equivalence class of a “strongly capped” endo-p-permutation kG-module under the compatibility relation.

1.4 The Brauer functor and the slash construction

Let G be a finite group, and let $H < K$ be subgroups of the group G. Let M be an $\mathcal{O}G$-module. The set M^H of H-fixed points in M is an $\mathcal{O}N_G(H)$-submodule of M. The relative trace map $\mathrm{Tr}_H^K : M^H \to M^K$ is defined by

$$\mathrm{Tr}_H^K(m) = \sum_{g \in K/H} gm \qquad \text{for any } m \in M.$$

The direct image $M_H^K = \mathrm{Tr}_H^K(M^H)$ is an $\mathcal{O}N_G(H,K)$-submodule of the module M^K of K-fixed points in M. If A is a G-algebra over the ring $\mathcal{O}$, then the direct image A_H^K of the relative trace map $\mathrm{Tr}_H^K : A^H \to A^K$ is an ideal of the subalgebra A^K of K-fixed points in A. If S is a normal subgroup of G and A is an S-interior G-algebra, the subalgebra A^K has a natural structure of $C_S(K)$-interior $N_G(K)$-algebra.

Let P be a p-subgroup of the group G, and M be an $\mathcal{O}G$-module. The Brauer quotient of M with respect to the p-subgroup P is defined by

$$\mathrm{Br}_P(M) \;=\; M^P/\big(\mathfrak{m}M^P + \sum_{Q<P} M_Q^P\big),$$

where $\mathfrak{m}$ is the maximal ideal of the local ring $\mathcal{O}$. The projection map $\mathrm{br}_P^M : M^P \to \mathrm{Br}_P(M)$ is called the Brauer map. We denote it with a lowercase b, while the Brauer quotient is denoted with a capital B. The quotient space $\mathrm{Br}_P(M)$ admits a unique structure of $kN_G(P)$-module such that the Brauer map $\mathrm{br}_P^M : M^P \to \mathrm{Br}_P(M)$ is an epimorphism of $\mathcal{O}N_G(P)$-modules. If S is a normal subgroup of G and A is an S-interior G-algebra, then the Brauer quotient $\mathrm{Br}_P(A)$ admits a unique structure of $C_S(P)$-interior $N_G(P)$-algebra such that the Brauer map $\mathrm{br}_P : A^P \to \mathrm{Br}_P(A)$ is an epimorphism of $C_S(P)$-interior $N_G(P)$-algebras.

Let L and M be two $\mathcal{O}G$-modules, and let $u : L \to M$ be a morphism of $\mathcal{O}P$-modules. Then u induces an $\mathcal{O}$-linear map $u^P : L^P \to M^P$, and there exists a unique k-linear map $\mathrm{Br}_P(u) : \mathrm{Br}_P(L) \to \mathrm{Br}_P(M)$ such that the following diagram commutes.

$$\begin{array}{ccc} L^P & \xrightarrow{\;u^P\;} & M^P \\ {\scriptstyle \mathrm{br}_P^L}\downarrow & & \downarrow{\scriptstyle \mathrm{br}_P^M} \\ \mathrm{Br}_P(L) & \xrightarrow[\mathrm{Br}_P(u)]{} & \mathrm{Br}_P(M) \end{array}$$

If $u : L \to M$ is a morphism of $\mathcal{O}G$-modules, then $\mathrm{Br}_P(u) : \mathrm{Br}_P(L) \to \mathrm{Br}_P(M)$ is a morphism of $kN_G(P)$-modules. If S is a normal subgroup of G and $u : A \to B$ is a morphism of S-interior G-algebras, then $\mathrm{Br}_P(u) : \mathrm{Br}_P(A) \to \mathrm{Br}_P(B)$ is a morphism of $C_S(P)$-interior $N_G(P)$-algebras. This defines a functor

$$\mathrm{Br}_P : {}_{\mathcal{O}G}\mathbf{Mod} \;\to\; {}_{kN_G(P)}\mathbf{Mod},$$

and a similar functor Br_P from the category of S-interior G-algebras to the category of $C_S(P)$-interior $N_G(P)$-algebras.

Let X be a G-set, and let $i : \mathcal{O}C_X(P) \to \mathcal{O}X$ be the inclusion map. The composition $\mathrm{br}_P^{\mathcal{O}X} \circ i : \mathcal{O}C_X(P) \to \mathrm{Br}_P(\mathcal{O}X)$ induces an isomorphism of $\mathcal{O}N_G(P)$-modules

$$kC_X(P) \;\simeq\; \mathrm{Br}_P(\mathcal{O}X).$$

This makes the Brauer functor a very nice tool to deal with p-permutation modules, as appears for example in [Rq-1998]. In particular, there is a natural isomorphism of $C_G(P)$-interior $N_G(P)$-algebras $kC_G(P) \simeq \mathrm{Br}_P(\mathcal{O}G)$, so we identify the Brauer map $\mathrm{br}_P^{\mathcal{O}G}$ with an epimorphism of $C_G(P)$-interior $N_G(P)$-algebras

$$\mathrm{br}_P : (\mathcal{O}G)P \;\to\; kC_G(P).$$

The Brauer functor is not as nice with endopermutation modules as it is with permutation module. The right tool for endopermutation modules is the slash construction, *a.k.a.* deflation-restriction, that has been defined in [Da-1978, Theorem 4.15]. Let P be a p-subgroup of the finite group G, and let M be an $\mathcal{O}G$-module such that the restriction $\mathrm{Res}^G_P M$ is an endopermutation $\mathcal{O}P$-module. Then the Brauer quotient $\mathrm{Br}_P(\mathrm{End}_{\mathcal{O}}(M))$ of the endomorphism algebra $\mathrm{End}_{\mathcal{O}}(M)$ is isomorphic to a (possibly zero) matrix algebra over the field k. In other words, there exists a k-vector space M' and an isomorphism

$$\mu : \mathrm{Br}_P(\mathrm{End}_{\mathcal{O}}(M)) \;\to\; \mathrm{End}_k(M').$$

The pair (M', μ) is unique up to an isomorphism that is itself unique up to scalar multiplication. We usually write $M(P) = M'$, and we call $M(P)$ a P-slashed module attached to M. The algebra $\mathrm{Br}_P(\mathrm{End}_{\mathcal{O}}(M))$ is, by construction, a $C_G(P)$-interior $N_G(P)$-algebra over the field k. This makes $M(P)$ a $kC_G(P)$-module. Furthermore, for any subgroup H of G such that $C_G(P) \leqslant H \leqslant N_G(P)$, it follows from the main result of [Pu-1986] that the $C_G(P)$-interior H-algebra $\mathrm{Br}_P(\mathrm{End}_{\mathcal{O}}(M))$ may be extended to an H-interior algebra. This extends the P-slashed module $M(P)$ to a kH-module. However, such an extension is hardly ever unique, as it can be twisted by any linear character $H/C_G(P) \to k^{\times}$. In general, there is no canonical choice among the possible extensions.

Although the correspondence $\mathrm{End}_{\mathcal{O}}(M) \mapsto \mathrm{End}_k(M(P))$ is functorial, it should be remembered that the slash construction, that we denote by $M \mapsto M(P)$, is *not* functorial in M. If the restriction $\mathrm{Res}_P^G M$ is a permutation $\mathcal{O}P$-module, then there is a natural isomorphism

$$\mathrm{Br}_P(\mathrm{End}_{\mathcal{O}}(M)) \;\simeq\; \mathrm{End}_k(\mathrm{Br}_P(M)).$$

In that case, the Brauer quotient $\mathrm{Br}_P(M)$ is thus a P-slashed module attached to M.

1.5 The local structure of finite groups and blocks

Let G be a finite group, and let e be a block of the algebra $\mathcal{O}G$. A subpair of the group G is a pair (P, e_P), where P is a p-subgroup of G and e_P is a block of the group algebra $\mathcal{O}C_G(P)$. The subpair (P, e_P) is an e-subpair if the reduction $\bar{e}_P \in kC_G(P)$ lies in the subalgebra $kC_G(P)\,\mathrm{br}_P(e) \simeq \mathrm{Br}_P(\mathcal{O}Ge)$, *i.e.*, if $\bar{e}_P\,\mathrm{br}_P(e) \neq 0$.

An e-subpair (P, e_P) of the group G is said to be a normal subpair of an e-subpair (Q, e_Q) if P is a normal subgroup of the p-group Q and $\bar{e}_Q\,\mathrm{br}_Q(e_P) \neq 0$; we write $(P, e_P) \lhd (Q, e_Q)$. An e-subpair (P, e_P) is contained in an e-subpair (Q, e_Q) if there is a chain of normal inclusions from the former to the latter; we write $(P, e_P) \leqslant (Q, e_Q)$. If (Q, e_Q) is an e-subpair of the group G and P is a subgroup of the p-group Q, then there is a unique block e_P of the group algebra $\mathcal{O}C_G(P)$ such that $(P, e_P) \leqslant (Q, e_Q)$.

The group G acts by conjugation on the set of e-subpairs of G. The set of maximal e-subpairs is an orbit under that action. A defect group of the block e is a p-subgroup D of G such that there exists a maximal e-subpair of the form (D, e_D). The set of defect groups of the block e is a conjugacy class of p-subgroups of G. More details on e-subpairs can be found in [AB-1979], which has settled the above definitions and notations.

The action of the group G on the lattice of e-subpairs can be described by the Brauer category $\mathbf{Br}(G, e)$. An object in $\mathbf{Br}(G, e)$ is an e-subpair (P, e_P). An arrow $\phi : (P, e_P) \to (Q, e_Q)$ is a group morphism $\phi : P \to Q$ such that there exists an element $g \in G$ that satisfies ${}^g(P, e_P) \leqslant (Q, e_Q)$ and $\phi(x) = {}^g x$ for any $x \in P$. The composition of arrows in $\mathbf{Br}(G, e)$ is the usual composition of group morphisms.

One case deserves to be singled out. The principal block e_0 of the group algebra $\mathcal{O}G$ is the unique block such that $e_0\mathcal{O} \neq 0$, where $\mathcal{O}$ stands for the principal $\mathcal{O}G$-module, *i.e.*, the free $\mathcal{O}$-module of rank 1 with trivial action of the group G. For any p-subgroup P of the group G, Brauer's third main theorem states the principal block f_0 of the algebra $\mathcal{O}C_G(P)$ is the only one such that the pair (P, f_0) is an e-subpair. In particular, it follows that a defect group of the principal block e_0 is a Sylow p-subgroup of the group G. Thus the Brauer category $\mathbf{Br}(G, e_0)$ actually describes the action of the group G on the lattice of its p-subgroups. This special Brauer category is called the Frobenius category of the group G, and we denote it by $\mathbf{Fr}(G)$.

Another approach is that of fusion systems and source algebras. For the next definitions and results, we follow [Li-2013]. Let (D, e_D) be a maximal e-subpair. A source idempotent of the block e with respect to the maximal subpair (D, e_D) is a primitive idempotent i of the algebra $(\mathcal{O}Ge)^D$ such that $\bar{e}_D \operatorname{br}_D(i) \neq 0$. The D-interior algebra $A = i\mathcal{O}Gi$ is called a source algebra of the block e. The $(A, \mathcal{O}Ge)$-bimodule $i\mathcal{O}G$ induces a Morita equivalence $A \sim \mathcal{O}Ge$.

For any subgroup P of D, the unique block e_P of the algebra $\mathcal{O}C_G(P)$ such that $(P, e_P) \leqslant (D, e_D)$ is characterised by the property $\bar{e}_Q \operatorname{br}_Q(i) \neq 0$. The fusion system of the block e with respect to the maximal subpair (D, e_D) is the category $\mathbf{F}$ defined as follows. An object of $\mathbf{F}$ is a subgroup P of D. If P, Q are two subgroups of D, an arrow $\phi : P \to Q$ is a group morphism such that there exists an element $g \in G$ that satisfies ${}^g(P, e_P) \leqslant (Q, e_Q)$ and $\phi(x) = {}^g x$ for any $x \in P$. There is an equivalence of categories $\mathbf{F} \to \mathbf{Br}(G, e)$, which sends a subgroup P of D to the e-subpair (P, e_P) such that $\bar{e}_P \operatorname{br}_P(i) \neq 0$, and a map $\phi : P \to Q$ to the same map ϕ, considered as an arrow $(P, e_P) \to (Q, e_Q)$.

The fusion system $\mathbf{F}$ can be read in the $(\mathcal{O}D, \mathcal{O}D)$-bimodule structure of the source $A = i\mathcal{O}Gi$. For any two subgroups P and Q of the defect group D, and any group morphism $\phi : P \to Q$, let ${}_\phi\mathcal{O}Q$ be the $\mathcal{O}$-module $\mathcal{O}Q$, considered as an $(\mathcal{O}P, \mathcal{O}Q)$-bimodule with the group P acting on the left via the map ϕ, and the group Q acting naturally on the right. The group morphism ϕ is an arrow in the fusion system $\mathbf{F}$ if, and only if, the $(\mathcal{O}P, \mathcal{O}Q)$-bimodule ${}_\phi\mathcal{O}Q$ is isomorphic to a direct summand of the restriction $\operatorname{Res}^{D\times D}_{P\times Q} A$.

The Brauer functor and the slash construction can easily be adapted to take subpairs into consideration. Let (P, e_P) be an e-subpair of the group G, and M be an $\mathcal{O}G$-module. We define the Brauer quotient of M with respect to the subpair (P, e_P) by

$$\mathrm{Br}_{(P,e_P)}(M) \;=\; \mathrm{Br}_P(e_P M).$$

The Brauer quotient $\mathrm{Br}_{(P,e_P)}(M)$ admits a unique structure of $kN_G(P, e_P)$-module such that the Brauer map $\mathrm{br}^M_{(P,e_P)} : M^P \to \mathrm{Br}_{(P,e_P)}(M)$, $m \mapsto \mathrm{br}^{e_P M}_P(e_P m)$, is an epimorphism of $\mathcal{O}N_G(P, e_P)$-modules. Let S be a normal subgroup of G such that the block e_P of the algebra $\mathcal{O}C_G(P)$ lies in the subalgebra $\mathcal{O}C_S(P)$. If A is an S-interior G-algebra, then we write $\mathrm{Br}_{(P,e_P)}(A) = \mathrm{Br}_P(e_P A e_P)$. This does not coincide with the Brauer quotient of the diagonal $\mathcal{O}G$-module A with respect to the subpair (P, e_P), but the context will usually make the notation unambiguous. The Brauer quotient $\mathrm{Br}_{(P,e_P)}(A)$ admits a unique structure of $C_S(P)$-interior $N_G(P, e_P)$-algebra such that the Brauer map $\mathrm{br}_{(P,e_P)} : A^P \to \mathrm{Br}_{(P,e_P)}(A)$, $a \mapsto \mathrm{br}^{e_P A e_P}_P(e_P a e_P)$ is an epimorphism of $C_S(P)$-interior $N_G(P, e_P)$-algebras.

Let L and M be two $\mathcal{O}G$-modules, and let $u : L \to M$ be a morphism of $\mathcal{O}P$-modules. Then the map u induces a morphism of $\mathcal{O}P$-modules $e_Q u e_Q : e_Q L \to e_Q M$, such that $(e_Q u e_Q)(x) = e_Q u(x)$ for any $x \in e_Q L$. We set $\mathrm{Br}_{(P,e_P)}(u) = \mathrm{Br}_P(e_P u e_P)$, a k-linear map from $\mathrm{Br}_{(P,e_P)}(L)$ to $\mathrm{Br}_{(P,e_P)}(M)$. If $u : L \to M$ is a morphism of $\mathcal{O}G$-modules, then $\mathrm{Br}_{(P,e_P)}(u)$ is a morphism of $kN_G(P, e_P)$-modules. This defines a functor

$$\mathrm{Br}_{(P,e_P)} : {}_{\mathcal{O}G}\mathbf{Mod} \;\to\; {}_{kN_G(P,e_P)}\mathbf{Mod}.$$

Similarly, we obtain a functor $\mathrm{Br}_{(P,e_P)}$ from the category of S-interior G-algebras to the category of $C_S(P)$-interior $N_G(P, e_P)$-algebras.

Let M be an $\mathcal{O}G$-module such that the direct summand $e_P \operatorname{Res}^G_P M$ is an endopermutation $\mathcal{O}P$-module. Then there exists a (P, e_P)-slashed module attached to M, *i.e.* a k-vector space $M(P, e_P)$ and an isomorphism

$$\mu : \mathrm{Br}_{(P,e_P)}(\mathrm{End}_{\mathcal{O}}(M)) \;\to\; \mathrm{End}_k(M(P, e_P)).$$

The pair $(M(P, e_P), \mu)$ is unique up to an isomorphism that is itself unique up to scalar multiplication. The algebra $\mathrm{Br}_{(P,e_P)}(\mathrm{End}_{\mathcal{O}}(M))$ is, by construction, a $C_G(P)$-interior $N_G(P, e_P)$-algebra over the field k. This makes $M(P, e_P)$ a $kC_G(P)$-module. Furthermore, for any subgroup H of G such that $C_G(P) \leqslant$

$H \leqslant N_G(P, e_P)$, the Brauer quotient $\mathrm{Br}_{(P,e_P)}(\mathrm{End}_{\mathcal{O}}(M))$ may be extended to an H-interior algebra. This extends the (P, e_P)-slashed module $M(P, e_P)$ to a kH-module, in a way that is only unique up to twisting by a linear character $H/C_G(P) \to k^\times$. If the restriction $e_P \operatorname{Res}^G_P M$ is a permutation $\mathcal{O}P$-module, then the Brauer quotient $\mathrm{Br}_{(P,e_P)}(M)$ naturally appears as a (P, e_P)-slashed module attached to M.

1.6 Fusion control

Let G be a finite group. Following [Pu-1976], we say that a subgroup H of G controls the p-fusion in the group G if the subgroup H has the same local structure as the group G, *i.e.*, if the inclusion map $H \to G$ induces an equivalence between the Frobenius categories $\mathbf{Fr}(H)$ and $\mathbf{Fr}(G)$. More explicitly, the subgroup H controls the p-fusion in the group G if there exists a Sylow p-subgroup D of G such that

- $D \leqslant H$;
- for any p-subgroup $P \leqslant D$ and any element $g \in G$ such that ${}^gP \leqslant D$, there exists $h \in H$ such that $h^{-1}g \in C_G(P)$.

More generally, let e be a block of the group algebra $\mathcal{O}G$, and (D, e_D) be a maximal e-subpair of G. We say that a subgroup H of G controls the e-fusion in the group G with respect to the maximal subpair (D, e_D) if

- $D \leqslant H$;
- for any e-subpair $(P, e_P) \leqslant (D, e_D)$ and any element $g \in G$ such that ${}^g(P, e_P) \leqslant (D, e_D)$, there exists $h \in H$ such that $h^{-1}g \in C_G(P)$.

Here, we need to specify the maximal e-subpair (D, e_D), because two maximal e-subpairs (D, e_D) and $(D', e_{D'})$ might not be H-conjugate, even though the defect groups D and D' are contained in H. In general, this definition cannot be rephrased in terms of Brauer categories, because there might not be a counterpart in the group algebra $\mathcal{O}H$ of the block e of the group algebra $\mathcal{O}G$. In any case, if H is a subgroup that controls the p-fusion in the group G, then *a fortiori* the subgroup H controls the e-fusion in the group G, for any block e of the algebra $\mathcal{O}G$.

If H is the centralizer of some p-subgroup P of G, then the subgroup $H =$

$C_G(P)$ controls the e-fusion in the group G if, and only if, the inclusion map $H \to G$ induces an equivalence between the Brauer categories $\mathbf{Br}(H, f)$ and $\mathbf{Br}(G, e)$, where f is any block of the algebra $\mathcal{O}H$ such that the pair (P, f) is an e-subpair.

Alperin's fusion theorem, proven in [Al-1967] for the fusion of p-subgroups and generalised in [AB-1979] to the fusion of e-subpairs, states that any isomorphism in the Brauer category $\mathbf{Br}(G, e)$ can be obtained by composing a sequence of isomorphisms of the form $\phi : (P, e_P) \to (Q, e_Q)$, $x \in P \mapsto {}^g x \in Q$, where g is an element of the normaliser $N_G(R, e_R)$ of an e-subpair (R, e_R) that contains both (P, e_P) and (Q, e_Q). Since any arrow in the Brauer category $\mathbf{Br}(G, e)$ is an isomorphism composed with an inclusion map, it follows that a subgroup H of G controls the e-fusion if, and only if, there exists a maximal e-subpair (D, e_D) of the group G such that

- $D \leqslant H$;
- $N_G(P, e_P) = C_G(P)\, N_H(P, e_P)$ for any subpair $(P, e_P) \leqslant (D, e_D)$.

Actually, there is a more precise result that restricts the second condition to the case where the subpair (P, e_P) is "essential", as defined in [Pu-1976]. We will not need this stronger version in the present thesis.

We now list a few classical results that give necessary or sufficient conditions for a subgroup to control fusion. Let G be a finite group that admits the factorisation

$$G \;=\; O_{p'}(G)\, H$$

for some subgroup H of the group G. Then the subgroup H controls the p-fusion in the group G (this will be proven as Fact 4.4). *A fortiori*, the subgroup H controls the e-fusion in the group G, for any block e of the algebra $\mathcal{O}G$.

There are at least two partial converses to the above result. The first one dates back to the beginning of the 20th century. If P is a p-subgroup of G that controls the p-fusion in the group G, then a theorem of Frobenius asserts that the group G admits the factorisation $G = O_{p'}(G)P$, *i.e.*, the group G is p-nilpotent. We will mention in Chapter 4 the generalisation of that theorem to nonprincipal blocks (remember that any statement about the p-fusion in a group G refers implicitly to the principal block e_0 of the algebra $\mathcal{O}G$).

A more recent result is the Z_p^*-theorem, which may be stated as follows.

$\mathbf{Z_p^*}$-theorem. *Let G be a finite group, and P be a p-subgroup of G. If the centraliser $C_G(P)$ controls the p-fusion in G, then the group G admits the factorisation*

$$G \;=\; O_{p'}(G)\, C_G(P).$$

This theorem was first proven in [Gl-1966] for $p = 2$. Glauberman's so-called Z^*-theorem was an important tool in the classification of finite simple groups, which was completed in the early 1980's. Then the classification made it possible to prove the Z_p^*-theorem for an odd prime p; a proof of the odd Z_p^*-theorem can be found in [Ar-1988, Theorem 1] or [GLS-1998, Remark 7.8.3].

The reader may find more information in the following references. [Br-1983] [GR-1998], [He-2004] and [Rw-1981] deal with the odd Z_p^*-theorem. [Wa-2013] gives a proof of the Z^*-theorem for $p = 2$, avoiding modular representations. Some of her techniques have analogs for odd primes. The unpublished paper [KO-2000] deals with a situation where the centraliser of a p-subgroup controls the fusion of a block. We will say more on this subject in Chapter 4.

1.7 Indecomposable modules

Let G be a finite group, and H be a subgroup of G. An $\mathcal{O}G$-module M is said to be relatively H-projective if it isomorphic to a direct summand of the induced module $\mathrm{Ind}_H^G L$ for some $\mathcal{O}H$-module L. Higman's criterion states that the $\mathcal{O}G$-module M is relatively H-projective if, and only if, the identity map $\mathbf{1}_M$ belongs to the direct image $\mathrm{End}_{\mathcal{O}}(M)_H^G$ of the relative trace map $\mathrm{Tr}_H^G : \mathrm{End}_{\mathcal{O}H}(M) \to \mathrm{End}_{\mathcal{O}G}(M)$. A very general version of that criterion can be found in [Br-2009].

Let M be an indecomposable $\mathcal{O}G$-module. A vertex of M is a subgroup P of G that is minimal with respect to the condition that the $\mathcal{O}G$-module M is relatively P-projective; in particular, the minimality assumption implies that P is a p-subgroup of G. An equivalent definition follows from Higman's criterion: a vertex of M is a p-subgroup P of G that is maximal with respect to the condition that the Brauer quotient $\mathrm{Br}_P(\mathrm{End}_{\mathcal{O}}(M))$ is nonzero.

A source of the indecomposable $\mathcal{O}G$-module M with respect to a given vertex P is an indecomposable $\mathcal{O}P$-module V such that M is isomorphic to a direct summand of the induced module $\operatorname{Ind}_P^G V$. Equivalently, a source of M with respect to the vertex P is any capped indecomposable direct summand of the restriction $\operatorname{Res}_P^G M$. The set of pairs $(P, [V])$, where P is a vertex of M and $[V]$ is an isomorphism class of source of M with respect to P, is an orbit under the natural action of the group G. As an example, a p-permutation $\mathcal{O}G$-module is a direct sum of indecomposable $\mathcal{O}G$-modules with trivial sources.

More details can be found in [Gr-1959], where Green also defines the following one-to-one correspondence.

Let M be an indecomposable $\mathcal{O}G$-module with vertex P and source V. A Green correspondent of M with respect to the vertex P is an indecomposable $\mathcal{O}N_G(P)$-module M' with vertex P such that the $\mathcal{O}G$-module M is isomorphic to a direct summand of the induced module $\operatorname{Ind}_{N_G(P)}^G M'$. Equivalently, a Green correspondent of M with respect to the vertex P is any indecomposable direct summand of the restriction $\operatorname{Res}_{N_G(P)}^G M$ that admits the p-group P as a vertex. The map $M \mapsto M'$ induces a one-to-one correspondence between the isomorphism classes of indecomposable $\mathcal{O}G$-modules with vertex P, and the isomorphism classes of indecomposable $\mathcal{O}N_G(P)$-modules with vertex P. An indecomposable $\mathcal{O}G$-module and its Green correspondent share the same set of sources with respect to their common vertex.

As an example, let e be a block of the group algebra $\mathcal{O}G$. Let (D, e_D) be a maximal e-subpair of the group G. The indecomposable p-permutation $\mathcal{O}(G \times G)$-module $\mathcal{O}Ge$ admits the diagonal p-subgroup ΔD as a vertex. Its Green correspondent with respect to that vertex is the $\mathcal{O}(C_G(D) \times C_G(D))\Delta N_G(D)$-module $\mathcal{O}C_G(D)f$, where $f = \operatorname{Tr}_{N_G(D,e_D)}^{N_G(D)} e_D$ is the unique idempotent in $Z(\mathcal{O}C_G(D))$ such that $\bar{f} = \operatorname{br}_D(e)$ in $kC_G(D)$.

We conclude this section with Nagao's theorem, which we take from [Be-1991, Theorem 6.3.1]. Let G be a finite group and e be a block of the algebra $\mathcal{O}G$. Let P be a p-subgroup of G, and let H be a subgroup of G such that $P \leqslant H \leqslant N_G(P)$. We denote by f the sum of the blocks e_P of $\mathcal{O}C_G(P)$ such that (P, e_P) is an e-subpair of G, *i.e.*, the unique idempotent in $Z(\mathcal{O}C_G(P))$ such that $\bar{f} = \operatorname{br}_P(e)$ in $kC_G(P)$. Thus the idempotent f is H-stable. For any $\mathcal{O}Ge$-module M, the restriction $\operatorname{Res}_H^G M$ decomposes as

$$\operatorname{Res}_H^G M \;=\; f\,M \;\oplus\; (1-f)\,M,$$

and no indecomposable direct summand of the $\mathcal{O}H$-module $(1-f)M$ admits a vertex that contains the p-group P. In particular, if M is an indecomposable $\mathcal{O}Ge$-module with vertex P, then its Green correspondent M', with respect to that vertex, is an $\mathcal{O}N_G(P)f$-module.

Chapitre 2
Équivalences de Morita et anneaux fortement gradués

Chapter 2
Morita equivalences and strongly graded rings

This chapter reviews the classical theory of Morita equivalences and strongly group-graded rings, with a slightly less classical approach based on the language of higher categories.

We describe Morita's theory of bimodules and functors between module categories, in terms of adjoint functors between 2-categories. For a given group G, we show that a strongly G-graded ring $R = \bigoplus_{g \in G} R_g$ may be considered as a ring $A = R_1$, endowed with an action of the group G on the category ${}_A\mathbf{Mod}$ of A-modules. We use this approach to study G-equivariant Morita equivalences and G-graded Morita equivalences. We obtain a sufficient condition for a Morita equivalence between block algebras to preserve the vertices of indecomposable modules. As an example, we give a new proof of the existence of a family of Morita equivalences that will be studied in more details in Chapter 4.

Throughout this chapter, we denote by $\mathcal{O}$ a commutative unitary ring. An $\mathcal{O}$-algebra is always assumed to be unitary, and an algebra morphism sends the unity of an algebra onto the unity of another algebra. We allow the case $\mathcal{O} = \mathbb{Z}$, where an $\mathcal{O}$-algebra is simply a unitary ring. When we need extra assumptions on the ring $\mathcal{O}$, we will make it clear.

Let R be an $\mathcal{O}$-algebra and G be a finite or infinite group. A G-grading on R is a family of $\mathcal{O}$-submodules $(R_g)_{g\in G}$, such that

$$R = \bigoplus_{g\in G} R_g \quad \text{and} \quad \forall (g,h) \in G^2, \quad R_g R_h \subseteq R_{gh}.$$

If two $\mathcal{O}$-algebras R and S admit G-gradings $(R_g)_{g\in G}$ and $(S_g)_{g\in G}$, then a morphism of G-graded algebras $f : R \to S$ is an algebra morphism such that $f(R_g) \subseteq S_g$ for any $g \in G$.

An $\mathcal{O}$-algebra R with a G-grading $(R_g)_{g\in G}$ is strongly G-graded if moreover $R_g R_h = R_{gh}$ for any g, h in G. The notion of a strongly G-graded algebra should be seen as the counterpart, in the category of algebras, of the notion of an extension of the group G in the category of groups. We denote by $\mathbf{SGAlg}(G, \mathcal{O})$ the category of strongly G-graded $\mathcal{O}$-algebras and graded algebra morphisms.

The theory of strongly group-graded algebras has been explored extensively in the last thirty years. We will refer to the work of Dade [Da-1980, Da-1981, Da-1982], Cohen and Montgomery [CM-1984], Külshammer [Kü-1985], Marcus [Ma-1996], among many others. As appears in [Da-1980], strongly group-graded algebras are a natural framework for the so-called Clifford theory. This application is what brings them in this thesis. We have in mind a more specific application, a Morita equivalence which is proven in Section 2.6 and will appear again in Chapter 4.

The present chapter is organised as follows. Section 2.1 deals with bimodules and functors between module categories. We define the bicategory $\mathbf{Bimod}_{\mathcal{O}}$ of $\mathcal{O}$-algebras, bimodules, and morphisms of bimodules, and the strict 2-category $\mathbf{ModCat}_{\mathcal{O}}$ of modules categories on $\mathcal{O}$-algebras, $\mathcal{O}$-linear functors, and natural transformations. We define a lax adjoint pair $(\mathcal{M}, \mathcal{M}^*)$ of functors between those bicategories. The adjunction property gathers, in one statement, many naturality and compatibility properties that are related to Morita's classical theorem, and that are needed in the remaining of this chapter. This section should be read as a dictionary, which we use to translate the theory of strongly group-graded algebras into a functorial language.

In Section 2.2, we set up our functorial approach to strongly group-graded algebras. For a group G and a strongly G-graded algebra $R = \bigoplus_{g\in G} R_g$, we check that the unit component R_1 is a subalgebra of R, and we show that there is a natural weak action of the group G on the category ${}_{R_1}\mathbf{Mod}$ of R_1-modules. We define a G-equivariant algebra as a pair $(A, \star)$ where A is an $\mathcal{O}$-algebra and $\star$ is an $\mathcal{O}$-linear weak action of the group G on the category ${}_A\mathbf{Mod}$. We prove that the category $\mathbf{EqAlg}(G, \mathcal{O})$ of G-equivariant algebras is equivalent to the category $\mathbf{SGAlg}(G, \mathcal{O})$ of strongly G-graded algebras. We illustrate with classical examples.

Section 2.3 is devoted to the study of G-invariant modules (also called extendible modules) over a G-equivariant algebra, which may be considered as modules over the corresponding G-graded algebra. For any two modules X and Y over a strongly G-graded algebra $R = \bigoplus_{g\in G} R_g$, we define the natural action of the group G on the $\mathcal{O}$-module of homomorphisms $\mathrm{Hom}_{R_1}(X, Y)$. As an application, we mention that relative traces and Higman's criterion can be used to deal with the relative projectivity of R-modules.

Section 2.4 is concerned with Morita equivalences. We define equivariant Morita equivalences between G-equivariant algebras, which are the counterpart of graded equivalences between strongly G-graded algebras. As a consequence of the group action defined in the previous section, we prove that any strongly G-graded algebra is Morita equivalent to a G-interior G-graded algebra. We also check that G-equivariant Morita equivalences preserve that group action, and we use Higman's criterion to deduce a result on vertex-preserving Morita equivalences.

The last two sections deal with examples. In Section 2.5, we use the second cohomology group $H^2(G, \mathcal{O}^\times)$ to classify all G-equivariant matrix algebras over a local ring $\mathcal{O}$, up to Morita equivalence. We deduce the classical theory of Clifford extensions for blocks of defect zero in normal subgroups. In Section 2.6, we give a new proof of the main result of [KR-1986], *i.e.* a Morita equivalence $kGe \sim kC_G(P)\,\mathrm{br}_P(e)$ for a finite group G, a p-subgroup P of G such that $G = O_{p'}(G)C_G(P)$, and a block e of the algebra kG such that $\mathrm{br}_P(e) \neq 0$.

2.1 Morita's theorem as a lax adjunction

For the definition of a bicategory, a 2-category and a lax functor, we follow [Le-2004]. We take the notion of a lax adjunction from [Bu-1974].

On the one hand, the bicategory $\mathbf{Bimod}_{\mathcal{O}}$ is defined as follows. An object is an $\mathcal{O}$-algebra A. For any two objects A and B, the category $\mathrm{Hom}_{\mathbf{Bimod}_{\mathcal{O}}}(A, B)$ is the category of (B, A)-bimodules and morphisms of bimodules. For any three objects A, B and C, the composition bifunctor is

$$- \otimes_B - \;:\; \mathrm{Hom}_{\mathbf{Bimod}_{\mathcal{O}}}(B, C) \times \mathrm{Hom}_{\mathbf{Bimod}_{\mathcal{O}}}(A, B) \;\longrightarrow\; \mathrm{Hom}_{\mathbf{Bimod}_{\mathcal{O}}}(A, C).$$

For any object A, the identity 1-arrow $\mathbf{1}_A : A \to A$ is the bimodule ${}_AA_A$. For any 1-arrow ${}_BM_A$, there are obvious 2-arrows $\lambda_M : \mathbf{1}_B \otimes_B M \to M$ and $\rho_M : M \otimes_A \mathbf{1}_A \to M$. For any three composable 1-arrows ${}_DN_C$, ${}_CM_B$ and ${}_BL_A$, there is an obvious 2-arrow $\alpha_{L,M,N} : (N \otimes_C M) \otimes_B L \to N \otimes_C (M \otimes_B L)$. The 2-arrows λ_M, ρ_M and $\alpha_{L,M,N}$ are invertible and natural in L, M, N. They satisfy the triangle and pentagon axioms.

On the other hand, we define a strict 2-category $\mathbf{ModCat}_{\mathcal{O}}$ as follows. An object is the $\mathcal{O}$-linear category ${}_A\mathbf{Mod}$ of left A-modules and morphisms of A-modules for some $\mathcal{O}$-algebra A. For any two objects ${}_A\mathbf{Mod}$ and ${}_B\mathbf{Mod}$, the category $\mathrm{Hom}_{\mathbf{ModCat}_{\mathcal{O}}}({}_A\mathbf{Mod}, {}_B\mathbf{Mod})$ is the category of $\mathcal{O}$-linear functors from ${}_A\mathbf{Mod}$ to ${}_B\mathbf{Mod}$. The composition of 1-arrows is the usual composition of functors, which is strictly associative and admits a strict identity $\mathbf{1}_{{}_A\mathbf{Mod}}$ for any object ${}_A\mathbf{Mod}$. By construction, $\mathbf{ModCat}_{\mathcal{O}}$ is a full 2-subcategory of the strict 2-category $\mathbf{Cat}$ of categories, functors and natural transformations.

We then consider the weak functor $\mathcal{M} : \mathbf{Bimod}_{\mathcal{O}} \to \mathbf{ModCat}_{\mathcal{O}}$ that sends an $\mathcal{O}$-algebra A to the $\mathcal{O}$-linear category ${}_A\mathbf{Mod}$, a bimodule ${}_BM_A$ to the $\mathcal{O}$-linear functor $M \otimes_A - : {}_A\mathbf{Mod} \to {}_B\mathbf{Mod}$, and a morphism $u : {}_BM_A \to {}_BM'_A$ to the natural transformation $u \otimes_A - : M \otimes_A - \to M' \otimes_A -$. For any object A in $\mathbf{Bimod}_{\mathcal{O}}$, there is an obvious natural map $\phi_A : \mathbf{1}_{{}_A\mathbf{Mod}} \to A \otimes_A -$, *i.e.*, a 2-arrow $\mathbf{1}_{\mathcal{M}A} \to \mathcal{M}\mathbf{1}_A$. For any composable 1-arrows ${}_CM_B$ and ${}_BL_A$, there is an obvious natural map

$$\phi_{L,M} : M \otimes_B (L \otimes_A -) \to (M \otimes_B L) \otimes_A -,$$

i.e., a 2-arrow $\mathcal{M}M \circ \mathcal{M}L \to \mathcal{M}(M \circ L)$. The 2-arrows ϕ_A and $\phi_{L,M}$ are invertible. They satisfy the naturality and consistency axioms of [Le-2004].

Finally, we consider the lax functor $\mathcal{M}^* : \mathbf{ModCat}_{\mathcal{O}} \to \mathbf{Bimod}_{\mathcal{O}}$ that sends an $\mathcal{O}$-linear category ${}_A\mathbf{Mod}$ to the $\mathcal{O}$-algebra A, an $\mathcal{O}$-linear functor $F : {}_A\mathbf{Mod} \to {}_B\mathbf{Mod}$ to the bimodule ${}_BF(A)_A$, and a natural transformation $\theta : F \to F'$ between functors $F, F' : {}_A\mathbf{Mod} \to {}_B\mathbf{Mod}$ to the morphism of bimodules $\theta(A) : {}_BF(A)_A \to {}_BF'(A)_A$. For any object ${}_A\mathbf{Mod}$ in $\mathbf{ModCat}_{\mathcal{O}}$, the identity morphism $\phi^*_{{}_A\mathbf{Mod}} : {}_AA_A \to {}_AA_A$ is a 1-arrow $\mathbf{1}_{\mathcal{M}^*{}_A\mathbf{Mod}} \to \mathcal{M}^*\mathbf{1}_{{}_A\mathbf{Mod}}$. For any composable 1-arrows $F : {}_A\mathbf{Mod} \to {}_B\mathbf{Mod}$ and $G : {}_B\mathbf{Mod} \to {}_C\mathbf{Mod}$, there is a morphism

$$\phi^*_{F,G} : {}_C[G(B) \otimes_B F(A)]_A \to {}_C[G \circ F(A)]_A,$$

i.e., a 2-arrow $\mathcal{M}^*G \circ \mathcal{M}^*F \to \mathcal{M}^*(G \circ F)$. The 2-arrows $\phi^*_{{}_A\mathbf{Mod}}$ and $\phi^*_{F,G}$ satisfy the same naturality and consistency axioms. However they are not invertible in general. The definition of the map $\phi^*_{F,G}$ relies on the *isomorphisme cher à Cartan*. We will make it explicit below.

It is quite clear that the weak functor $\mathcal{M}^*\mathcal{M}$ is isomorphic to the identity in $\mathbf{Bimod}_{\mathcal{O}}$. So the lax functor $\mathcal{M}^*$ behaves like a left inverse to the functor $\mathcal{M}$, and the lax functor $\mathcal{M}\mathcal{M}^*$ behaves like an idempotent in the 2-category $\mathbf{ModCat}_{\mathcal{O}}$. Morita's classical theorem states that a 1-arrow $F : {}_A\mathbf{Mod} \to {}_B\mathbf{Mod}$ in $\mathbf{ModCat}_{\mathcal{O}}$ is in the essential image of the idempotent endofunctor $\mathcal{M}\mathcal{M}^*$ if, and only if, F is a cocontinuous functor. This can be stated more precisely in terms of adjunctions.

Theorem. *The lax functor $\mathcal{M}^*$ is a (lax) right adjoint to the weak functor $\mathcal{M}$. In other words, there is a lax adjunction $(\mathcal{M}, \mathcal{M}^*, \eta, \varepsilon)$.*

The letter η stands for a natural transformation $\eta : \mathbf{1}_{\mathbf{Bimod}_{\mathcal{O}}} \to \mathcal{M}^*\mathcal{M}$. For each object A in $\mathbf{Bimod}_{\mathcal{O}}$, the bimodule $\eta_A = {}_AA_A$ is a 1-arrow $A \to \mathcal{M}^*\mathcal{M}A$. For each 1-arrow ${}_BM_A$, there is an obvious morphism of bimodules

$$\eta_M : {}_B[M \otimes_A A]_A \to {}_B[B \otimes_B (M \otimes_A A)]_A,$$

i.e., *a 2-arrow $\eta_B \circ M \to \mathcal{M}^*\mathcal{M}M \circ \eta_A$. The 1-arrow η_A and the 2-arrow η_M are (weakly) invertible and satisfy the three axioms of [Bu-1974, Definition 2.2].*

The letter ε stands for a lax natural transformation $\varepsilon : \mathcal{M}\mathcal{M}^* \to \mathbf{1}_{\mathbf{ModCat}_{\mathcal{O}}}$.

For each object ${}_A\mathbf{Mod}$ in $\mathbf{ModCat}_{\mathcal{O}}$, the identity functor $\varepsilon_{{}_A\mathbf{Mod}} = \mathbf{1}_{{}_A\mathbf{Mod}}$ is a 1-arrow $\mathcal{M}\mathcal{M}^*{}_A\mathbf{Mod} \to {}_A\mathbf{Mod}$. For each 1-arrow $F : {}_A\mathbf{Mod} \to {}_B\mathbf{Mod}$, there is a natural transformation

$$\varepsilon_F : F(A) \otimes_A - \to F(-),$$

i.e., *a 2-arrow $\varepsilon_{{}_B\mathbf{Mod}} \circ \mathcal{M}\mathcal{M}^* F \to F \circ \varepsilon_{{}_A\mathbf{Mod}}$. The map ε_F will be made explicit below. The 1-arrow $\varepsilon_{{}_A\mathbf{Mod}}$ and the 2-arrow ε_F satisfy the same axioms as above.*

However the 2-arrow $\varepsilon_F : F(A) \otimes_A - \to F(-)$ is not always invertible. Actually, ε_F is a natural isomorphism if and only if the functor $F : {}_A\mathbf{Mod} \to {}_B\mathbf{Mod}$ is cocontinuous, i.e., *commutes with small colimits. In particular, ε_F is a natural isomorphism whenever the functor F is an equivalence of categories.*

Finally, the quadruple $(\mathcal{M}, \mathcal{M}^*, \eta, \varepsilon)$ satisfies the axioms of [Bu-1974, Definition 3.1], which make it a lax adjunction.

The proof of the above theorem is essentially a good deal of straightforward diagram chasing. We do not go into details here. The only nontrivial part is the definition of the morphism of B-modules $\varepsilon_F(X) : F(A) \otimes_A X \to F(X)$ for a given functor $F : {}_A\mathbf{Mod} \to {}_B\mathbf{Mod}$ and a given A-module X. For the convenience of the reader, we now explain this construction.

On the one hand, there is an obvious morphism of A-modules $u : X \to \mathrm{Hom}_A(A, X)$ which sends $x \in X$ to the map $a \mapsto ax$. On the other hand, the functor F induces a morphism of A-modules $f : \mathrm{Hom}_A(A, X) \to \mathrm{Hom}_B(F(A), F(X))$. Then there is the *isomorphisme cher à Cartan*

$$c : \mathrm{Hom}_A(X, \mathrm{Hom}_B(F(A), F(X))) \longrightarrow \mathrm{Hom}_B(F(A) \otimes_A X, F(X)).$$

We set $\varepsilon_F(X) = c(f \circ u)$, a morphism of B-modules as required. As a by product, we obtain the definition of the morphism $\phi^*_{F,G} : G(B) \otimes_B F(A) \to G(F(A))$ that appears in the definition of the lax functor $\mathcal{M}^*$. For any two composable functors $F : {}_A\mathbf{Mod} \to {}_B\mathbf{Mod}$ and $G : {}_B\mathbf{Mod} \to {}_C\mathbf{Mod}$, we set $\phi^*_{F,G} = \varepsilon_G(F(A))$.

2.2 Strongly graded algebras and group actions

From this point on, we fix a group G. The bold letter $\mathbf{G}$ denotes the (strict) 2-category with one object $\centerdot$ where $\mathrm{End}_{\mathbf{G}}(\centerdot)$ is the discrete category defined on the set G and the composition functor $\mathrm{End}_{\mathbf{G}}(\centerdot) \times \mathrm{End}_{\mathbf{G}}(\centerdot) \to \mathrm{End}_{\mathbf{G}}(\centerdot)$ is derived from the group law in G. In other words, there is one object $\centerdot$ in the 2-category $\mathbf{G}$; a 1-arrow $\centerdot \to \centerdot$ is an element g of the group G, and the composition of 1-arrows is defined by $g \circ h = gh$ for any two elements g, h of G; the only 2-arrows are the identity 2-arrows attached to the 1-arrows, and the horizontal and vertical composition of 2-arrows are the only ones possible.

Definition 2.1. Let $\mathbf{C}$ be any bicategory and let X be an object in $\mathbf{C}$. A (weak) action $\star$ of the group G on the object X is a (weak) functor $\mathcal{A}^\star : \mathbf{G} \to \mathbf{C}$ such that $\mathcal{A}^\star(\centerdot) = X$. More explicitly, the action $\star$ is defined by

(D1) a 1-arrow $g_\star : X \to X$ for each $g \in G$,

(D2) an invertible 2-arrow $\theta_1^\star : \mathbf{1}_X \to 1_\star$,

(D3) an invertible 2-arrow $\theta_{g,h}^\star : g_\star \circ h_\star \to (gh)_\star$ for each $(g,h) \in G^2$,

such that the following two diagrams commute for any g, h, k in G:

$$\text{(A1)}\quad \begin{array}{ccccccc} \mathbf{1}_X \circ g_\star & & \xrightarrow{\lambda_{g_\star}} & g_\star & \xleftarrow{\rho_{g_\star}} & & g_\star \circ \mathbf{1}_X \\ & {\scriptstyle \theta_1^\star \circ \mathbf{1}_{g_\star}} \searrow & & {\scriptstyle \theta_{1,g}^\star} \nearrow \quad \nwarrow {\scriptstyle \theta_{g,1}^\star} & & \swarrow {\scriptstyle \mathbf{1}_{g_\star} \circ \theta_1^\star} & \\ & & 1_\star \circ g_\star & & g_\star \circ 1_\star & & \end{array} ,$$

$$\text{(A2)}\quad \begin{array}{ccccc} (g_\star \circ h_\star) \circ k_\star & & \xrightarrow{\alpha_{k_\star, h_\star, g_\star}} & & g_\star \circ (h_\star \circ k_\star) \\ \downarrow {\scriptstyle \theta_{g,h}^\star \circ \mathbf{1}_{k_\star}} & & & & \downarrow {\scriptstyle \mathbf{1}_{g_\star} \circ \theta_{h,k}^\star} \\ (gh)_\star \circ k_\star & \xrightarrow[\theta_{gh,k}^\star]{} & (ghk)_\star & \xleftarrow[\theta_{g,hk}^\star]{} & g_\star \circ (hk)_\star \end{array} .$$

Here the arrows $\lambda_{g_\star}$, $\rho_{g_\star}$, $\alpha_{k_\star, h_\star, g_\star}$ are the structure 2-arrows of the bicategory $\mathbf{C}$ (*cf.* the notations in the definition of the bicategory $\mathbf{Bimod}_{\mathcal{O}}$ in Section 2.1). If $\mathbf{C}$ were a strict 2-category, they would just be identity arrows. In particular, it follows from the definition that the 1-arrow $g_\star$ must be (weakly) invertible for any $g \in G$.

Before we can describe a strongly G-graded algebras in terms of a weak group action, we need a lemma.

Lemma 2.2. *Let $R = \bigoplus_{g\in G} R_g$ be a strongly G-graded algebra. Then R_1 is a subalgebra of R, and the multiplication map induces an isomorphism of (R_1, R_1)-bimodules $\mu_{g,h} : R_g \otimes_{R_1} R_h \to R_{gh}$ for any g, h in G.*

Proof. The first part of the statement is well known (one just needs to check that the unity 1_R lies in R_1). For the second part (which also appears in [Da-1980]), choose g, h in G. Setting $\mu_{g,h}(x \otimes y) = xy \in R_{gh}$, for any $x \in R_g$ and $y \in R_h$, clearly defines a morphism of (R_1, R_1)-bimodules $\mu_{g,h} : R_g \otimes_{R_1} R_h \to R_{gh}$. By assumption, $R_g R_h = R_{gh}$ so this morphism is surjective. In particular $R_{h^{-1}} R_h = R_1$ so there exists an element $\sum_i a_i \otimes b_i$ in $R_{h^{-1}} \otimes_{R_1} R_h$ such that $\sum_i a_i b_i = 1$ in R_1.

Define a map $\lambda_{g,h} : R_{gh} \to R_g \otimes_{R_1} R_h$ by setting $\lambda_{g,h}(z) = \sum_i (za_i) \otimes b_i$ for $z \in R_{gh}$. For any $z \in R_{gh}$, we have $\mu_{g,h} \circ \lambda_{g,h}(z) = \sum_i z a_i b_i = z \sum_i a_i b_i = z$, so that $\mu_{g,h} \circ \lambda_{g,h} = \mathrm{Id}_{R_{gh}}$. For any $x \in R_g$ and $y \in R_h$, we have $\lambda_{g,h} \circ \mu_{g,h}(x \otimes y) = \sum_i (xya_i) \otimes b_i$. Since $ya_i \in R_1$, this brings $\lambda_{g,h} \circ \mu_{g,h}(x \otimes y) = \sum_i x \otimes (ya_i b_i) = x \otimes \Big(y \sum_i a_i b_i\Big) = x \otimes y$. So we get $\lambda_{g,h} \circ \mu_{g,h} = \mathrm{Id}_{R_g \otimes_{R_1} R_h}$, which implies that $\mu_{g,h}$ is an isomorphism. □

Corollary 2.3. *Let A be an $\mathcal{O}$-algebra, and let $R = \bigoplus_{g\in G} R_g$ be a strongly G-graded $\mathcal{O}$-algebra endowed with an algebra isomorphism $\mu_1 : A \to R_1$. Then the (A, A)-bimodule R_g is a 1-arrow in $\mathbf{Bimod}_{\mathcal{O}}$ for any $g \in G$, $\mu_1 : \mathbf{1}_A \to R_1$ is an invertible 2-arrow and $\mu_{g,h} : R_g \otimes_A R_h \to R_{gh}$ is an invertible 2-arrow for any $g, h \in G$. Moreover these satisfy the axioms (A1) and (A2) of Definition 2.1. Thus they define a weak action of the group G on the object A of the bicategory $\mathbf{Bimod}_{\mathcal{O}}$.*

Conversely, let $\star$ be a weak action of the group G on the object A of the bicategory $\mathbf{Bimod}_{\mathcal{O}}$. Write $R^\star_g = \mathcal{A}^\star(g)$ for $g \in G$ and let $\mu^\star_1 : \mathbf{1}_A \to R_1$, $\mu^\star_{g,h} : R^\star_g \otimes_A R^\star_h \to R^\star_{gh}$ be the invertible 2-arrows associated with the action $\star$. Then the maps $\mu^\star_{g,h}$ define a multiplication law on the direct sum $R := \bigoplus_{g\in G} R_g$ which makes R a strongly G-graded $\mathcal{O}$-algebra. Moreover the map $\mu^\star_1$ is an algebra isomorphism from A onto the unit component R_1.

This motivates the following definition.

Definition 2.4. A G-equivariant algebra over the ring $\mathcal{O}$ is a pair $(A, \star)$ where A is an $\mathcal{O}$-algebra and $\star$ is an $\mathcal{O}$-linear weak action of the group G on the category ${}_A\mathbf{Mod}$. A morphism of G-equivariant algebras $(A, \star) \to (B, \bullet)$ is a pair (f, ψ), where $f : A \to B$ is a morphism of algebras and ψ is a family of natural isomorphisms $\psi_g : g_\star \circ \mathrm{Res}_f \to \mathrm{Res}_f \circ g_\bullet$ for $g \in G$ (where $\mathrm{Res}_f : {}_B\mathbf{Mod} \to {}_A\mathbf{Mod}$ is the functor defined by restriction through f), such that the following diagram commutes for any $g, h \in G$:

$$\text{(A3)} \quad \begin{array}{ccccc} g_\star \circ h_\star \circ \mathrm{Res}_f & \xrightarrow{\mathbf{1}_{g_\star} \circ \psi_h} & g_\star \circ \mathrm{Res}_f \circ h_\bullet & \xrightarrow{\psi_g \circ \mathbf{1}_{h_\bullet}} & \mathrm{Res}_f \circ g_\bullet \circ h_\bullet \\ \downarrow{\scriptstyle \theta^\star_{g,h} \circ \mathbf{1}_{\mathrm{Res}_f}} & & & & \downarrow{\scriptstyle \mathbf{1}_{\mathrm{Res}_f} \circ \theta^\bullet_{g,h}} \\ (gh)_\star \circ \mathrm{Res}_f & & \xrightarrow[\psi_{gh}]{\hspace{6em}} & & \mathrm{Res}_f \circ (gh)_\bullet \end{array}$$

We denote by $\mathbf{EqAlg}(G, \mathcal{O})$ the category of G-equivariant $\mathcal{O}$-algebras and morphisms of G-equivariant $\mathcal{O}$-algebras.

Let $(A, \star)$ be a G-equivariant $\mathcal{O}$-algebra. The action $\star$ is defined by a weak functor $\mathcal{A}^\star : \mathbf{G} \to \mathbf{ModCat}_\mathcal{O}$ such that $\mathcal{A}^\star(\centerdot) = {}_A\mathbf{Mod}$. Then the weak functor $\mathcal{M}^* \circ \mathcal{A}^\star : \mathbf{G} \to \mathbf{Bimod}_\mathcal{O}$ defines an action of the group G on the object A in $\mathbf{Bimod}_\mathcal{O}$. By Corollary 2.3, this in turn defines a strongly G-graded algebra $R = \bigoplus_{g \in G} R_g$ together with an algebra isomorphism $A \simeq R_1$. Explicitly, we have $R_g = g_\star A$ for any $g \in G$, and the multiplication map $R \otimes_A R \to R$ is defined by linearity from the composed morphisms

$$g_\star A \otimes_A h_\star A \xrightarrow{\phi^*_{h_\star, g_\star}} g_\star(h_\star A) \xrightarrow{\theta^\star_{g,h}} (gh)_\star A.$$

for g, h in G. We call R the twisted product of the algebra A by the group G relative to the action $\star$, and we denote it by $A_\star G$.

Let $(f, \psi) : (A, \star) \to (B, \bullet)$ be a morphism of G-equivariant algebras. Then $f : A \to \mathrm{Res}^B_A B$ is a morphism of left A-modules. We define a map $\tilde{f}_g : g_\star A \to g_\bullet B$ by composing the morphisms of left A-modules

$$g_\star A \xrightarrow{g_\star f} g_\star \mathrm{Res}_f B \xrightarrow{\psi_g(B)} \mathrm{Res}^B_A g_\bullet B.$$

By linearity, the maps $\tilde{f}_g$, $g \in G$, induce a map $\tilde{f} : A_\star G \to B_\bullet G$ which is a morphism of G-graded algebras.

Theorem 2.5. *The correspondences $(A, \star) \mapsto A_\star G$, $(f, \phi) \mapsto \tilde{f}$ define an equiv-*

alence of categories

$$\mathcal{G} : \mathbf{EqAlg}(G, \mathcal{O}) \to \mathbf{SGAlg}(G, \mathcal{O}).$$

Proof. The functoriality is straightforward. The functor $\mathcal{G}$ is essentially surjective because of the natural isomorphism $\eta : \mathbf{1}_{\mathbf{Bimod}_{\mathcal{O}}} \to \mathcal{M}^*\mathcal{M}$.

Let $(A, \star)$ be a G-equivariant $\mathcal{O}$-algebras. The functor $g_\star : {}_A\mathbf{Mod} \to {}_A\mathbf{Mod}$ is an equivalence of categories for any $g \in G$; in particular, it is cocontinuous. So the natural morphism $\varepsilon_{g_\star} : g_\star A \otimes_A - \to g_\star$ of the Morita theorem is an isomorphism, and it follows that the functor $\mathcal{G}$ is fully faithful. $\square$

A G-equivariant algebra should be seen as a generalisation of a (twisted) group action on an algebra. We give a few examples.

Example 2.6. Let A be an $\mathcal{O}$-algebra and $\gamma : G \to \mathrm{Aut}_{\mathcal{O}\text{-}\mathbf{Alg}}(A)$ an action of the group G on the algebra A, so that the pair (A, γ) is a G-algebra. For $g \in G$ and X an A-module, denote by $g_\star X$ the twisted module defined as follows. As an $\mathcal{O}$-module, $g_\star X = X$ but it is endowed with the action of A defined by $(a, x) \mapsto \gamma(g^{-1})(a) \cdot x$. If $u : X \to Y$ is a morphism of A-modules, the map u is also a morphism of A-modules from $g_\star X$ to $g_\star Y$ for any $g \in G$. By setting $g_\star u = u$, we define an $\mathcal{O}$-linear functor $g_\star : {}_A\mathbf{Mod} \to {}_A\mathbf{Mod}$. We then have $1_\star = \mathbf{1}_{{}_A\mathbf{Mod}}$ and $(gh)_\star = g_\star \circ h_\star$ for any g, h in G, so that we can choose $\theta^\star_1 = \mathrm{Id} : 1_\star \to \mathbf{1}_{{}_A\mathbf{Mod}}$ and $\theta^\star_{g,h} = \mathrm{Id} : g_\star \circ h_\star \to (gh)_\star$. These data define an $\mathcal{O}$-linear action $\star$ of the group G on the category ${}_A\mathbf{Mod}$, and make $(A, \star)$ a G-equivariant algebra.

More generally, let $S \triangleleft G$ be a normal subgroup, and let $\gamma : G \to \mathrm{Aut}_{\mathcal{O}\text{-}\mathbf{Alg}}(A)$ and $\iota : S \to A^\times$ be two group morphisms such that $\gamma(s)(a) = \iota(s) a \, \iota(s)^{-1}$ and $\iota(gsg^{-1}) = \gamma(g)(\iota(s))$ for any $a \in A$, $s \in S$ and $g \in G$. The triple (A, γ, ι) is thus an S-interior G-algebra. Denote the quotient group G/S by $\bar{G}$. For each element $\bar{g} \in \bar{G}$, choose a preimage $r_{\bar{g}} \in G$, and denote by $\bar{g}_\star$ the functor $(r_{\bar{g}})_\star$, as defined above. For any two elements $\bar{g}, \bar{h} \in \bar{G}$, there is a unique $s \in S$ such that $r_{\bar{g}} r_{\bar{h}} = r_{\bar{g}\bar{h}} s$. Then the multiplication by $\iota(s) \in A^\times$ defines a natural isomorphism $\theta^\star_{\bar{g},\bar{h}} : \bar{g}_\star \circ \bar{h}_\star \to (\bar{g}\bar{h})_\star$, and the multiplication by $\iota(r_{\bar{1}})$ defines a natural isomorphism $\theta^\star_1 : \bar{1}_\star \to \mathrm{Id}_{{}_A\mathbf{Mod}}$. These data define an $\mathcal{O}$-linear action $\star$ of the group $\bar{G}$ on the category ${}_A\mathbf{Mod}$. Up to natural isomorphism, this action does not depend on the choice of the preimages $r_{\bar{g}}$.

The S-interior G-algebra A is naturally an $\mathcal{O}(S \times S)\Delta G$-module, where ΔG denotes the diagonal subgroup of $G \times G$. There is a natural isomorphism $A_\star \bar{G} \simeq \mathrm{Ind}_{(S\times S)\Delta G}^{G\times G} A$. In particular, the twisted product $A_\star \bar{G}$ is a G-interior $\bar{G}$-graded algebra. When $S = 1$, we recover the usual smash product $A\sharp G$ of A by G.

An important example is the following. Let G be a finite group, S be a normal subgroup of G and b be a G-stable block of the group algebra $\mathcal{O}S$. Then the conjugation action of the group G makes $\mathcal{O}Sb$ an S-interior G-algebra, hence a G/S-equivariant algebra. The twisted product $(\mathcal{O}Sb)_\star(G/S)$ is naturally isomorphic to the $\bar{G}$-graded algebra $\mathcal{O}Gb$.

Example 2.7. Let $\alpha : G\times G \to \mathcal{O}^\times$ be a 2-cocycle. For $g \in G$, let $g_\alpha : {}_{\mathcal{O}}\mathbf{Mod} \to {}_{\mathcal{O}}\mathbf{Mod}$ be the identity functor. For g, h in G, let $\theta^\alpha_{g,h} : (gh)_\alpha \to g_\alpha \circ h_\alpha$ be the multiplication by $\alpha(g, h)^{-1}$. Let $\theta^\alpha_1 : 1_\alpha \to \mathrm{Id}_{{}_{\mathcal{O}}\mathbf{Mod}}$ be the multiplication by $\alpha(1, 1)^{-1}$. These data define an $\mathcal{O}$-linear action of the group G on the category ${}_{\mathcal{O}}\mathbf{Mod}$. With a slight abuse, we will denote the associated G-equivariant algebra by $(\mathcal{O}, \alpha)$. The twisted product $\mathcal{O}_\alpha G$ is isomorphic to the usual twisted group algebra of G over the ring $\mathcal{O}$ associated with the 2-cocycle α.

It is easily checked that two 2-cocycles α and β define isomorphic actions of G on ${}_{\mathcal{O}}\mathbf{Mod}$ if, and only if, they are cohomologous. So the isomorphism class of the G-equivariant algebra $(\mathcal{O}, \alpha)$ depends only on the cohomology classes $[\alpha]$ in $H^2(G, \mathcal{O}^\times)$.

If moreover the ring $\mathcal{O}$ is local, then any $\mathcal{O}$-linear equivalence from the category ${}_{\mathcal{O}}\mathbf{Mod}$ to itself is isomorphic to the identity functor. As a consequence, every G-equivariant structure on the algebra $\mathcal{O}$ derives from a 2-cocycle α, up to isomorphism. Thus we have defined a one-to-one correspondence between the cohomology classes $[\alpha] \in H^2(G, \mathcal{O}^\times)$ and the isomorphism classes of G-equivariant algebras $(A, \star)$ such that $A \simeq \mathcal{O}$.

2.3 Graded modules and invariant modules

Let $R = \bigoplus_{g\in G} R_g$ be a G-graded algebra over the ring $\mathcal{O}$, and let X be an R-module. A G-grading on X is a family of $\mathcal{O}$-submodules $(X_g)_{g\in G}$, such that

$$X = \bigoplus_{g\in G} X_g \quad \text{and} \quad \forall (g,h) \in G^2, \quad R_g X_h \subseteq X_{gh}.$$

If two R-modules X and Y admit G-gradings $(X_g)_{g\in G}$ and $(Y_g)_{g\in G}$, then a morphism of G-graded R-modules $u : X \to Y$ is an algebra morphism such that $u(X_g) \subseteq Y_g$ for any $g \in G$. The following result is [Da-1980, Theorem 2.8].

Lemma 2.8. *Let $(A,\star)$ be a G-equivariant $\mathcal{O}$-algebra, and $A_\star G$ be the corresponding strongly G-graded $\mathcal{O}$-algebra. For any A-module X, the induced $A_\star G$-module $X' = A_\star G \otimes X$ admits the G-grading $X' = \bigoplus_{g\in G} X'_g$, where $X'_g = g_\star A \otimes_A X$ for $g \in G$. In particular, there is a natural isomorphism $X'_1 \simeq X$. This construction defines an equivalence between the category ${}_A\mathbf{Mod}$ of A-modules, and the category ${}_{A_\star G}\mathbf{GrMod}$ of G-graded $A_\star G$-modules.*

We now introduce another important notion. A G-invariant A-module is a pair (X,χ), where X is an A-module and χ is a family of isomorphisms $\chi_g : g_\star X \to X$ for $g \in G$, such that the following diagram commutes for any g, h in G:

(A4)
$$\begin{array}{ccc} g_\star h_\star X & \xrightarrow{\theta^\star_{g,h}(X)} & (gh)_\star X \\ \downarrow{\scriptstyle g_\star\chi_h} & & \downarrow{\scriptstyle \chi_{gh}} \\ g_\star X & \xrightarrow[\chi_g]{} & X \end{array} .$$

We may also say that the family $\chi = (\chi_g)_{g\in G}$ is a G-invariant structure on the A-module X. Let (X,χ) and (Y,ψ) be two G-invariant A-modules. A morphism of A-modules $u : X \to Y$ is said to be G-invariant if $u \circ \chi_g = \psi_g \circ g_\star u$ for any $g \in G$.

By calling the pair (X,χ) a G-invariant module, we follow [Ta-2001]. In the specific context of strongly graded algebras, [Da-1981] calls X a G-invariant A-module if there exists an isomorphism $g_\star X \simeq X$ for each $g \in G$, and an extendible A-module if there exists a consistent family of isomorphisms $(\chi_g)_{g\in G}$ as above (the latter condition being clearly stronger than the former). The following result is [Da-1980, Theorem 2.8, Theorem 2.12]. It is also very close to [Ta-2001, Theorem 3.1], in a different context.

Lemma 2.9. *Let* $(A, \star)$ *be a* G*-equivariant algebra and* (X, χ) *be a* G*-invariant* A*-module. The* A*-module* X *extends naturally to an* $A_\star G$*-module. This construction defines an equivalence between the category* $({}_A\mathbf{Mod})^G$ *of* G*-invariant* A*-modules and* G*-invariant* A*-linear maps, and the category* ${}_{A_\star G}\mathbf{Mod}$ *of all* $A_\star G$*-modules.*

Proof. For the convenience of the reader, we explain the extension of X to an $A_\star G$-module. If (X, χ) is a G-invariant A-module, then we have the isomorphism $\chi_g : g_\star X \to X$. We compose it with the isomorphism $\varepsilon_{g_\star}(X) : g_\star A \otimes_A X \to g_\star X$ of Morita's theorem (*cf.* Section 2.1) to obtain a map $g_\star A \otimes_A X \to X$ for any $g \in G$. Since $A_\star G = \bigoplus_{g \in G} g_\star A$, these induce by linearity a map $A_\star G \otimes_A X \to X$. The naturality and compatibility conditions on the action $\star$ and the invariant structure χ imply that the latter map makes X an $A_\star G$-module. The equivalence of categories is proven in either of the above references. □

Let $(A, \star)$ be a G-equivariant algebra and (X, χ) be a G-invariant A-module. The G-invariant structure χ on the A-module X induces a structure of G-algebra on the endomorphism algebra $\mathrm{End}_A(X)$. Indeed, for any element g of the group G and any endomorphism u of the A-modules X, we let $\gamma_g(u)$ be the unique endomorphism of the A-module X which makes the following diagram commutative; remember that χ_g in an isomorphism of A-modules:

$$\begin{array}{ccc} g_\star X & \xrightarrow{\;g_\star u\;} & g_\star X \\ {\scriptstyle \chi_g}\downarrow & & \downarrow{\scriptstyle \chi_g} \\ X & \xrightarrow[\gamma_g(u)]{\cdots\cdots} & X \end{array}$$

The functoriality of $g_\star$ implies $\gamma_g(v \circ u) = \gamma_g(v) \circ \gamma_g(u)$ for any two endomorphisms u, v of X. Thus the map γ_g is an automorphism of the $\mathcal{O}$-algebra B. Moreover, it follows from the commutativity of Diagram (A4) that $\gamma_g \circ \gamma_h = \gamma_{gh}$ for any two elements g, h in G. So the map $g \mapsto \gamma_g$ is a group morphism $G \to \mathrm{Aut}_{\mathbf{Alg}}(\mathrm{End}_A(X))$ and makes $\mathrm{End}_A(X)$ a G-algebra.

Lemma 2.10. *Let* $(A, \star)$ *be a* G*-equivariant algebra and* X *be an* $A_\star G$*-module. By Lemma 2.9, there is a natural* G*-invariant structure on the* A*-module* X *which makes* $\mathrm{End}_A(X)$ *a* G*-algebra. Moreover* $\mathrm{End}_A(X)^G = \mathrm{End}_{A_\star G}(X)$.

Proof. By Lemma 2.9, an endomorphism of the $A_\star G$-module X is a G-invariant endomorphism of the A-module X, *i.e.*, precisely a fixed point of the action of the group G on the algebra $\mathrm{End}_A(X)^G$. This also appears in [Da-1982, Theorem 2.1]. □

With the notations of Lemma 2.10, let $H \leqslant K$ be two subgroups of G. Then we can define a relative trace map $\mathrm{Tr}_H^K : \mathrm{End}_{A_\star H}(X) \to \mathrm{End}_{A_\star K}(X)$ by setting $\mathrm{Tr}_H^K(u) = \sum_{g \in K/H} \gamma_g(u)$. Thus Higman's criterion may be used to deal with relative projectivity in the category of modules over a G-graded algebra, as appears in [Da-1982, Proposition 3.3].

2.4 Morita equivalences

Let $(A, \star)$ and $(B, \bullet)$ be two G-equivariant $\mathcal{O}$-algebras. On the one hand, a G-invariant (or G-equivariant) functor ${}_A\mathbf{Mod} \to {}_B\mathbf{Mod}$ is a pair (F, ψ) where $F : {}_A\mathbf{Mod} \to {}_B\mathbf{Mod}$ is a functor and ψ is a family of natural isomorphisms $\psi_g : g_\star \circ F \to F \circ g_\bullet$ for $g \in G$, such that a diagram similar to (A3) commutes for any $g, h \in G$.

On the other hand, we define a weak action $\vartriangle$ of the group G on the category ${}_{A\otimes B^{\mathrm{op}}}\mathbf{Mod}$ of (A, B)-bimodules by considering the functor

$$g_\vartriangle = g_\star - \otimes_B g_\bullet^{-1} B$$

for any $g \in G$, and the natural isomorphisms $\theta_1^\vartriangle = \theta_1^\star(-) \otimes_B \theta_1^\bullet(B)$ and $\theta_{g,h}^\vartriangle = \theta_{g,h}^\star(-) \otimes_B \theta_{h^{-1},g^{-1}}^\bullet(B)$ for any g, h in G. This makes the pair $(A \otimes B^{\mathrm{op}}, \vartriangle)$ a G-equivariant algebra. We will identify the strongly G-graded algebra $(A \otimes B^{\mathrm{op}})_\vartriangle G$ to the subalgebra $\bigoplus_{g \in G} g_\star A \otimes (g_\bullet^{-1} B)^{\mathrm{op}}$ of the tensor product $(A_\star G) \otimes (B_\bullet G)^{\mathrm{op}}$. By Lemma 2.9, a G-invariant (A, B)-bimodule may be seen as an $(A \otimes B^{\mathrm{op}})_\vartriangle G$-module.

With these definitions, we can define a notion of G-equivariant Morita equivalence.

Theorem 2.11. *Let $(A, \star)$, $(B, \bullet)$ be G-equivariant algebras over the ring $\mathcal{O}$. The following three conditions are equivalent :*

(i) there is a G-invariant equivalence of categories ${}_B\mathbf{Mod} \to {}_A\mathbf{Mod}$;

(ii) *there are a G-invariant (A, B)-bimodule M and a (B, A)-bimodule N such that*

$$\begin{aligned} M \otimes_B N &\simeq A && \text{as } (A, A)\text{-bimodules,} \\ N \otimes_A M &\simeq B && \text{as } (B, B)\text{-bimodules;} \end{aligned}$$

(iii) *there are a G-graded $(A_\star G, B_\bullet G)$-bimodule M' and a $(B_\bullet G, A_\star G)$-bimodule N' such that*

$$\begin{aligned} M' \otimes_{B_\bullet G} N' &\simeq A_\star G && \text{as } (A_\star G, A_\star G)\text{-bimodules,} \\ N' \otimes_{A_\star G} M' &\simeq B_\bullet G && \text{as } (B_\bullet G, B_\bullet G)\text{-bimodules.} \end{aligned}$$

If these conditions are satisfied, we say that there is a Morita equivalence $(A, \star) \sim (B, \bullet)$, or a G-equivariant Morita equivalence $A \sim B$, or a G-graded Morita equivalence $A_\star G \sim B_\bullet G$.

Proof. Suppose that $(F, \psi) : {}_B\mathbf{Mod} \to {}_A\mathbf{Mod}$ is a G-equivariant equivalence of categories. Choose a functor $F' : {}_A\mathbf{Mod} \to {}_B\mathbf{Mod}$ and natural isomorphisms $\alpha : F \circ F' \to \mathbf{1}_{{}_A\mathbf{Mod}}$ and $\beta : F' \circ F \to \mathbf{1}_{{}_B\mathbf{Mod}}$. Consider the (A, B)-bimodule $M = F(B) = \mathcal{M}^*(F)$ and $N = F'(A) = \mathcal{M}^*(F')$ with their respective natural structure of (B, A)-bimodule. Since F and F' are cocontinuous functors, the maps $\phi^*_{F,F'} : M \otimes_B N \to F \circ F'(A)$ and $\phi^*_{F',F} : N \otimes_A M \to F' \circ F(B)$ defined in Section 2.1 are isomorphisms of bimodules. By composing them with the isomorphism $\alpha(A) : F \circ F'(A) \to A$ and $\beta(B) : F' \circ F(B) \to B$, we obtain the isomorphisms of *(ii)*.

We now focus on the (A, B)-bimodule $g_\vartriangle M = g_\star M \otimes_B g_\bullet^{-1} B$ for some $g \in G$. We define an isomorphism of (A, B)-bimodules $\chi_g : g_\vartriangle M \to M$ by composing the isomorphisms

$$\begin{array}{ccccc} g_\vartriangle M = g_\star(F(B)) \otimes_B g_\bullet^{-1} B & & \xrightarrow{\quad \phi^*_{g_\star \circ F, g_\bullet^{-1}} \quad} & & [g_\star \circ F \circ g_\bullet^{-1}](B) \\ \Big\downarrow{\scriptstyle \chi_g} & & & & \Big\downarrow{\scriptstyle \psi_g(g_\bullet^{-1} B)} \\ F(B) = M & \xleftarrow{\ F(\theta^\bullet_1(B))\ } & [F \circ 1_\bullet](B) & \xleftarrow{\ F(\theta^\bullet_{g,g^{-1}}(B))\ } & [F \circ g_\bullet \circ g_\bullet^{-1}](B) \end{array}$$

It follows from all the above naturality and compatibility conditions that the pair (M, χ) is a G-invariant $A \otimes B^{\mathrm{op}}$-module. This proves *(i)* $\Rightarrow$ *(ii)*. The reverse implication is proven in the same way, by using the weak functor $\mathcal{M}$ instead of $\mathcal{M}^*$.

The implication *(ii)* $\Rightarrow$ *(iii)* is essentially [Ma-1996, Theorem 3.4]. Conversely, suppose that the statement in *(ii)* is satisfied, for a G-graded $(A_\star G, B_\bullet G)$-bimodule $M' = \bigoplus_{g\in G} M'_g$. As we know from Morita theory, we may suppose that $N' = \mathrm{Hom}_{A_\star G}(M', A)$ is the $A_\star G$-dual of M'. Thus $N' = \bigoplus_{g\in G} N'_g$ with $N'_g \simeq \mathrm{Hom}_A(M'_g, A)$, and N' is also a G-graded bimodule. Then the component M'_1 is an $(A \otimes B^{\mathrm{op}})_\vartriangle G$-module, *i.e.*, a G equivariant (A, B)-bimodule. Moreover, the isomorphisms of Proposition *(iii)* induce isomorphisms of bimodules $M'_1 \otimes_B N'_1 \simeq A$ and $N'_1 \otimes_A M'_1 \simeq B$. This proves *(iii)* $\Rightarrow$ *(ii)* and the theorem. □

The category of $(A\otimes B^{\mathrm{op}})_\vartriangle G$-modules and the category of G-graded $(A_\star G \otimes B_\bullet G)$-modules are equivalent (the precise statement and its proof are very similar to those of Lemma 2.8). So the proof of the theorem defines an equivalence of categories between G-equivariant Morita equivalences $A \sim B$ and G-graded Morita equivalences $A_\star G \sim A_\bullet G$.

As an application, we can reduce the problem of classifying all strongly G-graded algebras up to graded Morita equivalence to that of classifying all G-algebras up to G-equivariant Morita equivalence. The following result is essentially a corollary of [CM-1984, Theorem 2.12].

Theorem 2.12. *For any strongly G-graded algebra R, there is a G-graded Morita equivalence $R \sim S$, where S is a G-interior G-graded algebra. For any G-equivariant algebra $(A, \star)$, there is a G-equivariant Morita equivalence $A \sim B$, where B is a G-algebra.*

Proof. The first statement follows immediately from the second one by Theorem 2.11. So we consider a G-equivariant algebra $(A, \star)$. For any $g \in G$, the functor $g_\star : {}_A\mathbf{Mod} \to {}_A\mathbf{Mod}$ is an equivalence of categories, so the left A-module $g_\star A$ is a progenerator in the abelian category ${}_A\mathbf{Mod}$. As a consequence, the left A-module $A_\star G$ is a progenerator in ${}_A\mathbf{Mod}$.

More generally, let X be a left $A_\star G$-module whose restriction to A is a progenerator in ${}_A\mathbf{Mod}$. We know from Theorem 2.9 that there exists a G-invariant structure χ on the A-module X. From Section 2.3 we have an action of the group G on the algebra $\mathrm{End}_A(X)$, hence on the opposite algebra $B =$

$\mathrm{End}_A(X)^{\mathrm{op}}$. Denote by $(B, \bullet)$ the G-equivariant algebra associated with the G-algebra B as in Example 2.6. As X is a progenerator in the category ${}_A\mathbf{Mod}$, the (A, B)-bimodule X provides a Morita equivalence $A \sim B$.

For any $g \in G$, we have by definition $g_\vartriangle X = g_\star X \otimes_B g^{-1}_\bullet B$; remember that $g^{-1}_\bullet B$ is just B as a set. We define a map $\tilde{\chi}_g : g_\vartriangle X \to X$ by setting $\tilde{\chi}(x \otimes 1_B) = \chi_g(x)$ for any $x \in X$. By construction of the maps $\gamma_g : B \to B$, the maps $\tilde{\chi}_g$ are isomorphisms of (A, B)-bimodules and satisfy the commutativity of a diagram similar to (A4). So the pair $(X, \tilde{\chi})$ is a G-invariant (A, B)-bimodule and provides a Morita equivalence $(A, \star) \sim (B, \bullet)$. □

We now prove that G-equivariant Morita equivalences preserve the structure of G-algebra on the endomorphism algebras of G-invariant modules.

Lemma 2.13. *Let $(A, \star)$ and $(B, \bullet)$ be two G-equivariant algebras, and let (F, ψ) be a G-invariant functor from the category ${}_A\mathbf{Mod}$ to ${}_B\mathbf{Mod}$. Consider a G-invariant A-module (X, χ) and the corresponding G-invariant B-module $(F(X), \chi')$. The functor F induces a morphism of G-algebras $\mathrm{End}_A(X) \to \mathrm{End}_B(F(X))$.*

Proof. Let us define explicitly the natural G-invariant structure on the B-module $F(X)$. For $g \in G$, we define the map $\chi'_g : g_\bullet F(X) \to F(X)$ by composing the isomorphisms of B-modules

$$g_\bullet F(X) \xrightarrow{\psi_g(X)} F(g_\star X) \xrightarrow{F(\chi_g)} F(X).$$

This provides the pair $(F(X), \chi')$. The rest of the proof is just diagram checking. □

As a consequence, G-graded Morita equivalences preserve a certain amount of relative projectivity. With additional hypotheses, we can obtain a more precise result. From now on, we let $\mathcal{O}$ be a complete discrete valuation ring with positive residual characteristic p.

For $i = 1, 2$, we let G_i be a finite group, S_i be a normal subgroup of G_i, and b_i be a G_i-stable block of the group algebra $\mathcal{O}S_i$. Moreover we consider a finite group G with a normal subgroup S, together with group morphisms $f_i : G \to G_i$ such that $f_i^{-1}(S_i) = S$ and $S_i f_i(G) = G_i$ for $i = 1, 2$. We set $\bar{G} = G/S$, so that the morphisms f_1, f_2 induce isomorphisms $\bar{G} \simeq G_1/S_1 \simeq G_2/S_2$. In particular, there are natural $\bar{G}$-gradings on the algebras $\mathcal{O}G_1b_1$ and $\mathcal{O}G_2b_2$ which make them strongly $\bar{G}$-graded algebras.

Theorem 2.14. *In addition to the above notation, suppose that there exists a subgroup H of G such that $S \cap H = 1$ and that $f_i(H)$ contains a defect group of the primitive G_i-algebra $\mathcal{O}S_ib_i$ for $i = 1, 2$. Then any $\bar{G}$-graded Morita equivalence $\mathcal{O}G_1b_1 \sim \mathcal{O}G_2b_2$ preserves vertices in the following sense : for any indecomposable $\mathcal{O}G_1b_1$-module X_1 with vertex $Q_1 \leqslant f_1(H)$, the corresponding indecomposable $\mathcal{O}G_2b_2$-module X_2 admits the vertex $Q_2 = f_2(H \cap f_1^{-1}(Q_1))$.*

Proof. Let $M' = \bigoplus_{g\in\bar{G}} M'_g$ be a $\bar{G}$-graded $(\mathcal{O}G_2b_2, \mathcal{O}G_1b_1)$-bimodule which provides a Morita equivalence $\mathcal{O}G_2b_2 \sim \mathcal{O}G_1b_1$. The unit component $M = M'_1$ is a $\bar{G}$-invariant $(\mathcal{O}S_2b_2, \mathcal{O}S_1b_1)$-bimodule which provides a Morita equivalence $\mathcal{O}S_2b_2 \sim \mathcal{O}S_1b_1$. Let X_1 be an $\mathcal{O}G_1b_1$-module, hence a $\bar{G}$-invariant $\mathcal{O}S_1b_1$-module. By Lemma 2.13, the $\mathcal{O}S_2b_2$-module $X_2 = M \otimes_{\mathcal{O}S_1b_1} X_1$ has a natural $\bar{G}$-invariant structure, which also makes it a $\mathcal{O}G_2b_2$-module. As such, X_2 is naturally isomorphic to $M' \otimes_{\mathcal{O}G_1b_1} X_1$.

We now suppose that X_1 is indecomposable as an $\mathcal{O}G_1b_1$-module and X_2 as an $\mathcal{O}G_2b_2$-module, and we study their vertices. We set $H_i = f_i(H)$ for $i = 1, 2$; we have in particular $S_i \cap H_i = 1$. By hypothesis, the idempotent $b_1 \in (\mathcal{O}S_1)^{G_1}$ is relatively (H_1, G_1)-projective so the indecomposable kG_1b_1-module X_1 is relatively D_1-projective. As a consequence, it admits a vertex Q_1 which is contained in H_1. As $Q_1 \leqslant S_1Q_1$, we deduce from Higman's criterion the existence of an element $u \in \mathrm{End}_{\mathcal{O}}(X_1)^{S_1Q_1}$ such that $\mathrm{Tr}^{G}_{S_1Q_1}(u) = \mathbf{1}_{X_1}$.

The $\bar{G}$-invariant structure on the $\mathcal{O}S_1b_1$-module X_1 induces an action of the group $\bar{G}$ on the endomorphism algebra $\mathrm{End}_{\mathcal{O}S_1b_1}(X_1)$. Set $Q = f_1^{-1}(S_1Q_1) \cap H$, a p-subgroup of G that the map f_1 sends isomorphically onto Q_1, and set $\bar{Q} = SQ/S \simeq S_1Q_1/S_1$, a p-subgroup of $\bar{G}$. Then u is an element in $\mathrm{End}_{\mathcal{O}S_1b_1}(X_1)^{\bar{Q}}$ such that $\mathrm{Tr}^{\bar{G}}_{\bar{Q}}(u) = \mathbf{1}_{X_1}$.

We deduce from Lemma 2.13 that $v = \mathbf{1}_M \otimes u$ is an element in $\mathrm{End}_{\mathcal{O}S_2b_2}(X_2)^{\bar{Q}}$ such that $\mathrm{Tr}^{\bar{G}}_{\bar{Q}}(v) = \mathbf{1}_{X_2}$. Set $Q_2 = f_2(Q)$; remember that $Q \leqslant H$ so that $Q_2 \leqslant H_2$ and f_2 induces an isomorphism $\bar{Q} \simeq S_2Q_2/S_2$. Then v is an element in $\mathrm{End}_{\mathcal{O}}(X_2)^{S_2Q_2}$ such that $\mathrm{Tr}^{G_2}_{S_2Q_2}(v) = \mathbf{1}_{X_2}$.

By hypothesis, the idempotent b_2 is relatively (H_2, G_2)-projective. Let $D_2 \leqslant H_2$ be a defect group of the primitive idempotent b_2 in the group G_2; then D_2 is maximal among the Sylow p-subgroups of the centralisers in G_2 of the elements of S_2 which support $b_2 \pmod{p\mathcal{O}}$. With this characterisation, it is clear that D_2 is again a defect group of the primitive idempotent b_2 in the group S_2H_2 of the primitive idempotent $b_2 \in (\mathcal{O}S_2b_2)^{S_2D_2}$. Thus b_2 is relatively (H_2, SH_2)-projective: there exists an $a \in (\mathcal{O}S_2b_2)^{H_2}$ such that $b_2 = \mathrm{Tr}_{H_2}^{S_2H_2}(a)$. We now apply the Mackey formula for relative traces, keeping in mind that there is only one double class in $S_2Q_2\backslash S_2H_2/H_2$, and that $S_2Q_2 \cap H_2 = Q_2$.

$$\begin{aligned} \mathbf{1}_{X_2} &= b_2\,\mathrm{Tr}_{S_2Q_2}^{G_2}(v) \;=\; \mathrm{Tr}_{S_2H_2}^{G_2}\Big(\mathrm{Tr}_{H_2}^{S_2H_2}(a)\,.\,\mathrm{Tr}_{S_2Q_2}^{S_2H_2}(v)\Big) \\ &= \mathrm{Tr}_{S_2H_2}^{G_2}\Big(\sum_{g\in S_2Q_2\backslash S_2H_2/H_2} \mathrm{Tr}_{S_2Q_2\cap gH_2g^{-1}}^{S_2D_2}(gag^{-1}.v)\Big) \;=\; \mathrm{Tr}_{Q_2}^{G_2}(av). \end{aligned}$$

Thus the indecomposable $\mathcal{O}G_2b_2$-module X_2 is relatively (Q_2, G_2)-projective, and it has a vertex contained in Q_2. Since the p-group Q_2 is isomorphic to Q_1, it follows in particular that the order of a vertex of X_2 is less than or equal to the order of a vertex of X_1. But the roles of X_1 and X_2 are symmetric here, so a vertex of X_2 must have the same order as a vertex of X_1. We conclude that Q_2 is a vertex of X_2. □

Notice that Theorem 2.14 still holds when the ring $\mathcal{O}$ is replaced by the field k. In the next two sections, we give examples of Morita equivalences which preserve vertices.

2.5 Matrix algebras and Clifford extensions

The following proposition classifies all G-equivariant algebras $(A, \star)$ such that A is isomorphic to a matrix algebra over a local ring $\mathcal{O}$. The correspondence that we obtain is essentially the classical theory of Clifford extensions.

Proposition 2.15. *Suppose that $\mathcal{O}$ is a local ring, and consider a G-equivariant algebra $(A, \star)$ such that A is isomorphic to a matrix algebra over the ring $\mathcal{O}$. Then $(A, \star)$ is Morita equivalent to $(\mathcal{O}, \alpha)$ for a unique cohomology class $[\alpha] \in H^2(G, \mathcal{O}^\times)$, where $(\mathcal{O}, \alpha)$ is the G-equivariant algebra defined in Example 2.7.*

Let $(A, \star)$ and $(B, \bullet)$ be G-equivariant matrix algebras over $\mathcal{O}$, and let $[\alpha]$ and $[\beta]$ be the associated with cohomology classes. There is a Morita equivalence $(A, \star) \sim (B, \bullet)$ if, and only if, one has $[\alpha] = [\beta]$ in the cohomology group

$H^2(G, \mathcal{O}^\times)$. Moreover, the tensor product $(A \otimes B^{\mathrm{op}}, \vartriangle)$ is a G-equivariant matrix algebra, and the associated cohomology class is $[\delta] = [\alpha][\beta]^{-1}$.

Proof. Since A is isomorphic with a matrix algebra $M_n(\mathcal{O})$, there exists an equivalence of categories $F : {}_{\mathcal{O}}\mathbf{Mod} \to {}_{A}\mathbf{Mod}$. Then we can define by pullback an $\mathcal{O}$-linear weak action $\bullet$ of the group G on the category ${}_{\mathcal{O}}\mathbf{Mod}$ such that the functor F admits a G-invariant structure. When the $\mathcal{O}$ is local, we have proven in Example 2.7 that the G-equivariant algebra $(\mathcal{O}, \bullet)$ must derive from a cohomology class $[\alpha] \in H^2(G, \mathcal{O}^\times)$.

It remains to prove that the class $[\alpha]$ is uniquely defined. Let $\alpha, \beta : G \times G \to \mathcal{O}^\times$ be cocycles such that the G-equivariant algebras $(\mathcal{O}, \alpha)$ and $(\mathcal{O}, \beta)$ are Morita equivalent. Any bimodule which provides a Morita equivalence $\mathcal{O} \sim \mathcal{O}$ is isomorphic to $\mathcal{O}$ itself, so the G-equivariant algebras $(\mathcal{O}, \alpha)$ and $(\mathcal{O}, \beta)$ are actually isomorphic. It is then easy to check that the cocycles α and β are cohomologous, which proves the requested unicity.

For the last statement, we choose a G-invariant $(A, \mathcal{O})$-bimodule M which provides a Morita equivalence $(A, \star) \sim (\mathcal{O}, \alpha)$ and a G-invariant $(\mathcal{O}, B)$-bimodule N which provides a Morita equivalence $(\mathcal{O}, \beta) \sim (B, \bullet)$. Thus $M \otimes N$ is a G-invariant $(A \otimes B^{\mathrm{op}}, \mathcal{O})$-bimodule which provides a Morita equivalence $(A \otimes B^{\mathrm{op}}, \vartriangle) \sim (\mathcal{O}, \delta)$. □

As a typical application, let $\mathcal{O}$ be a complete discrete valuation ring; we suppose that its residue field k has positive characteristic p, and that its fraction field $\mathbb{K}$ is big enough for the finite goups which we consider. Let G be a finite group, S be a normal subgroup, and b be a G-stable block of defect zero of the group algebra $\mathcal{O}S$. Then, as in Example 2.6, the block algebra $\mathcal{O}Sb$ is a G/S-equivariant algebra which corresponds to the strongly G/S-graded algebras kGb. Let $[\alpha] \in H^2(G/S, \mathcal{O}^\times)$ be the associated cohomology class. By Proposition 2.15, there is a $\bar{G}$-equivariant Morita equivalence $\mathcal{O}Sb \sim (\mathcal{O}, \alpha)$. Thus by Theorem 2.11, there is a $\bar{G}$-graded Morita equivalence $\mathcal{O}Gb \sim \mathcal{O}_\alpha(G/S)$. We could apply Theorem 2.14 to prove that this Morita equivalence preserves vertices, a fact already proven in [Ha-2011].

2.6 Dade algebras and the Brauer functor

In this section, we replace the ring $\mathcal{O}$ with an algebraically closed field k of positive characteristic p. Let P be a p-group. Remember from Section 1.3 that a Dade P-algebra is a P-algebra A over the field k such that A is isomorphic to a matrix algebra and admits a P-stable basis with at least one element fixed by P (this element can actually be chosen to be the unity 1_A).

Lemma 2.16. *Let G be a finite group, P be a p-subgroup of G, and S be a normal subgroup of G such that $S \cap P = 1$. Let A be an S-interior G-algebra over the field k such that $\mathrm{Res}^G_P(A)$ is a Dade P-algebra. Then the algebras A and $\mathrm{Br}_P(A)$ are Morita equivalent as $N_G(P)/C_S(P)$-equivariant algebras.*

Proof. Since A is a Dade P-algebra and the field k is algebraically closed, we know from [Th-1995] that the Brauer quotient $B = \mathrm{Br}_P(A)$ is a nonzero matrix algebra. Moreover, B has a natural structure of $C_S(P)$-interior $N_G(P)$-algebra. Notice that the condition $S\cap P = 1$ implies that $N_S(P) = C_S(P)$. By restriction, A is also a $C_S(P)$-interior $N_G(P)$-algebra. Thus both A and B have natural structures of $\bar{G}$-equivariant algebras, where $\bar{G} = N_G(P)/C_S(P)$.

Since A and B are matrix algebra over k, the same is true for the tensor product $F = A\otimes B^{\mathrm{op}}$. Let M be a simple (A, B)-bimodule, so that the structure map $F \to \mathrm{End}_k(M)$ is an isomorphism. Then M provides a Morita equivalence $A \sim B$. To fit Condition *(ii)* of Theorem 2.11, all we need now is to extend the bimodule M to an $(A \otimes B^{\mathrm{op}})_\Delta\bar{G}$-module.

The quotient SP/S is a p-group and A is a matrix algebra over the algebraically closed field k of characteristic p, so there is a unique extension of A to an SP-interior G-algebra (this follows from the Skolem-Noether theorem). Then, on the one hand, the SP-interior G-algebra A has a natural structure of $k(SP \times SP)\Delta G$-module, which makes the endomorphism ring $\mathrm{End}_k(A)$ an $(SP \times SP)\Delta G$-interior algebra. On the other hand, the tensor product $E = A \otimes A^{\mathrm{op}}$ is an $(SP \times SP)$-interior $(G \times G)$-algebra. Moreover A is a matrix algebra over k, so the structure map of the (A, A)-bimodule A is an isomorphism $\lambda : E \to \mathrm{End}_k(A)$. Clearly λ is an isomorphism of $SP \times SP$-interior $(SP\times SP)\Delta G$-algebras. By pullback, we extend E to an $(SP\times SP)\Delta G$-interior algebra.

We now consider the p-subgroup $(1\times P)$ of $(SP\times SP)\Delta G$. Its centraliser contains $(C_S(P)\times C_S(P))\Delta C_G(P)$, and its normaliser contains $(C_S(P)\times C_S(P))\Delta N_G(P)$. Thus the Brauer quotient $\mathrm{Br}_{1\times P}(E)$ is naturally a $(C_S(P)\times C_S(P))\Delta C_G(P)$-interior $(C_S(P)\times C_S(P))\Delta N_G(P)$-algebra. By [Pu-1986, (d)], it can be extended to a $(C_S(P)\times C_S(P))\Delta N_G(P)$-interior algebra. Moreover the obvious map $F\to \mathrm{Br}_{1\times P}(E)$ is clearly an isomorphism of $(C_S(P)\times C_S(P))$-interior $(C_S(P)\times C_S(P))\Delta N_G(P)$-algebras. By pullback, we can then extend F to a $(C_S(P)\times C_S(P))\Delta N_G(P)$-algebra.

We denote by $\iota : (C_S(P)\times C_S(P))\Delta N_G(P)\to F$ the interior structure map. The strongly graded algebra $(A\otimes B^{\mathrm{op}})_{\vartriangle}\bar{G}$ also has a natural $(C_S(P)\times C_S(P))\Delta N_G(P)$-interior structure; actually, it is even generated by $A\otimes B^{\mathrm{op}}$ and the elements of $\Delta N_G(P)$ (see Example 2.6). Thus the interior structure map ι enables us to extend the identity map $A\otimes B^{\mathrm{op}}\to F$ to an algebra morphism $(A\otimes B^{\mathrm{op}})_{\vartriangle}\bar{G}\to F$. As a result, the F-module M is extended to a $\bar{G}$-equivariant (A,B)-bimodule. This completes the proof. □

Theorem 2.17. *Let G be a finite group. Let S be a normal subgroup of G and P a p-subgroup of G such that $S\cap P=1$. Suppose that $G=SN_G(P)$. Let b be a block of defect zero of the algebra kS such that P centralises b and $\mathrm{br}_P(b)\neq 0$ (the latter condition is unnecessary if S is a p'-group). Let $b'\in kS$ be the sum of all G-conjugates of b. Then the algebras kGb' and $kN_G(P)\,\mathrm{br}_P(b')$ are Morita equivalent.*

In particular, if e is any block of kG that covers b, then the algebras kGe and $kN_G(P)\,\mathrm{br}_P(e)$ are Morita equivalent. Moreover, if X is an indecomposable kGe-module and Y is the corresponding $kN_G(P)\,\mathrm{br}_P(e)$-module, then any vertex of Y is also a vertex of X.

Proof. We denote by G_b the stabiliser of b in the group G, and by $\bar{G}$ the quotient $N_{G_b}(P)/C_S(P)$; the condition $G=SN_G(P)$ implies $\bar{G}\simeq G_b/S$. We consider the S-interior G_b-algebra $A=kSb$, which restricts to a Dade P-algebra. By Lemma 2.16, there is a $\bar{G}$-equivariant Morita equivalence $A\sim \mathrm{Br}_P(A)$, *i.e.*, $kSb\simeq kC_S(P)\,\mathrm{br}_P(b)$. By Theorem 2.11, this induces a $\bar{G}$-graded Morita equivalence $kG_bb\sim kN_{G_b}(P)\,\mathrm{br}_P(b)$.

Let D be a defect group in G_b of the primitive idempotent $b\in (kS)^{G_b}$. Since $\mathrm{br}_P(b)\neq 0$, we may suppose that the p-group D contains P. The subgroup SP is normal in G, so D must normalise the intersection $SP\cap D=P$. Thus D is also a defect group of the primitive idempotent $\mathrm{br}_P(b)\in (kC_S(P))^{N_{G_b}(P)}$. Therefore we can apply Theorem 2.14 with $H = D$, and deduce that the $\bar{G}$-graded Morita equivalence $kG_b b \sim kN_{G_b}(P)\,\mathrm{br}_P(b)$ preserves vertices.

It is then classic (see for instance [Ha-2007, Proposition 1.2]) that the induction/restriction functors induce vertex-preserving Morita equivalences $kGb' \sim kG_b b$ and $kN_{G_b}(P)\,\mathrm{br}_P(b) \sim kN_G(P)\,\mathrm{br}_P(b')$. This completes the proof. □

Remark 2.18. Theorem 2.17 is essentially the main theorem of [KR-1986]. We have slightly improved it by lifting the hypothesis that S is a p'-group, and replacing $C_G(P)$ by $N_G(P)$. This improved version was already known to Dade, Puig and Harris, with proofs relying on Clifford extensions. We have stated and proven the theorem over a field k of characteristic p, but it is not very difficult to lift the result to a complete discrete valuation ring with residue field k. The main tool is [Ca-1987, Proposition 2]. However, we will not need this lifting in the present thesis.

Corollary 2.19. *Let G be a finite group and e a block of G with defect group D. Suppose that there exist a normal subgroup S of G such that $S \cap D = 1$, and a subgroup P of D such that $G = SN_G(P)$. Then there is a vertex-preserving Morita equivalence $kGe \sim kN_G(P)\,\mathrm{br}_P(e)$.*

Proof. This is straightforward from Theorem 2.17, thanks to [Kn-1976, Proposition 4.2]. □

Chapitre 3
Modules et catégories compatibles avec la localisation

Chapter 3
Brauer-friendly modules and categories

The Brauer functor is a very nice tool to handle p-permutation modules, i.e., *modules with trivial sources. In this chapter, we construct a similar tool that is fit for modules with endopermutation sources.*

We say that a category ${}_{\mathcal{O}Ge}\mathbf{M}$ *of modules over a block algebra* $\mathcal{O}Ge$ *is "Brauer-friendly" if all indecomposable direct summands of objects in this category admit fusion-stable endopermutation sources that are compatible with one another. Given such a category, and an e-subpair* (P, e_P) *of the group* G*, we define a "slash functor"* $\mathrm{Sl}_{(P,e_P)} : {}_{\mathcal{O}Ge}\mathbf{M} \to {}_{kN_G(P,e_P)\bar{e}_P}\mathbf{Mod}$*, which satisfies most of the properties of the Brauer functor* $\mathrm{Br}_{(P,e_P)} : {}_{\mathcal{O}Ge}\mathbf{Perm} \to {}_{kN_G(P,e_P)\bar{e}_P}\mathbf{Mod}$*. In particular, we obtain a handy parametrisation of the indecomposable modules that belong to a given Brauer-friendly category.*

Throughout this chapter, we fix a prime number p and a p-modular system $(\mathbb{K}, \mathcal{O}, k)$. We remind the reader that we allow the case $\mathcal{O} = k$, so that any statement that has been proven over the complete discrete valuation ring $\mathcal{O}$ remains true over the residue field k. The notations and definitions that we use have been set up in Chapter 1.

Let us recall some of the very nice properties of the Brauer functor, when it is applied to p-permutation modules. The following statement gathers some of those properties, the proofs of which can be found in [Br-1985] and [Rq-1998].

Fact 3.1. *Let G be a finite group.*

(i) For any p-subgroup P of G, and any two p-permutation $\mathcal{O}G$-modules L and M, there is a natural isomorphism of $k(C_G(P) \times C_G(P))\Delta N_G(P)$-modules

$$\mathrm{Br}_P(\mathrm{Hom}_{\mathcal{O}}(L, M)) \;\to\; \mathrm{Hom}_k(\mathrm{Br}_P(L), \mathrm{Br}_P(M)).$$

(ii) For any two p-subgroups P and Q of G such that $P \lhd Q$, and any p-permutation $\mathcal{O}G$-module M, there is a natural isomorphism of $kN_G(P,Q)$-modules

$$\mathrm{Br}_Q(M) \;\to\; \mathrm{Br}_Q \circ \mathrm{Br}_P(M).$$

(iii) For any p-subgroup P of G, any element $g \in G$, and any p-permutation $\mathcal{O}G$-module M, there is a natural isomorphism of $kN_G(P)$-modules

$$\mathrm{Br}_P(M) \;\to\; g^{-1}\,\mathrm{Br}_{{}^gP}(M).$$

(iv) The map $M \mapsto \mathrm{Br}_P(M)$ induces a one-to-one correspondence between the isomorphism classes of indecomposable p-permutation $\mathcal{O}G$-modules with vertex P and the isomorphism classes of projective indecomposable $kN_G(P)/P$-modules.

These properties usually fail as soon as the Brauer functor is used to deal with modules with non-trivial sources. In this chapter, we aim at defining a class of $\mathcal{O}G$-modules and a construction relative to a p-subgroup P of the group G that satisfies all the properties listed above. When the group G is a p-group, this question has been partially answered in [Da-1978]. Dade's slash construction for endopermutation modules lacks functoriality, but it complies with Fact 3.1 (ii), (iii), and even (i) if L and M are compatible endopermutation modules.

A possible generalisation of endopermutation modules to an arbitrary finite group G is the class of endo-p-permutation $\mathcal{O}G$-module, which have been studied by Urfer in his PhD thesis [Ur-2006]. The slash construction equally applies to those modules. However, endo-p-permutation modules do not seem to show up very often in the context of Morita or stable equivalences, except over the principal block. In any case, they are not enough for the application that we have in mind, *i.e.*, the main result of Chapter 4.

A further-reaching generalisation, due to Puig, needs to be formulated in terms of source algebras. Let G be a finite group, e be a block of the algebra $\mathcal{O}G$, D be a defect group of the block e, and $i \in (\mathcal{O}G)^D$ be a source idempotent for the block e with respect to the block e. Let $A = i\mathcal{O}Gi$ be the corresponding source algebra, which is by construction a D-interior algebra. The mapping $M \mapsto iM$ induces an equivalence of categories ${}_{\mathcal{O}Ge}\mathbf{Mod} \to {}_{A}\mathbf{Mod}$, so that working over the source algebra A is essentially the same as working over the block algebra $\mathcal{O}Ge$. Let M be a $\mathcal{O}Ge$-module such that the A-module iM is an endopermutation $\mathcal{O}D$-module (via the D-interior structure of the source algebra A). Then the slash construction can be applied to the module iM, with the same nice properties as above, and the same lack of functoriality.

The purpose of this chapter is to bring back the slash construction to the level of the block algebra $\mathcal{O}Ge$, for the class of $\mathcal{O}Ge$-modules such that the corresponding A-modules are endopermutation $\mathcal{O}D$-modules. We call those $\mathcal{O}Ge$-modules "Brauer-friendly". Then we wish to make the slash construction functorial. This cannot happen on the category of all Brauer-friendly $\mathcal{O}Ge$-modules, because two such modules are not compatible in general (*i.e.*, the corresponding A-modules are not compatible as endopermutation $\mathcal{O}D$-modules). Thus we restrain to what we call "Brauer-friendly" subcategories of ${}_{\mathcal{O}Ge}\mathbf{Mod}$, one of which is the category ${}_{\mathcal{O}Ge}\mathbf{Perm}$ of p-permutation $\mathcal{O}Ge$-modules, and we define "slash functors" separately for each of these subcategories.

This collection of subcategories might encourage one to go a step further, and define a "Dade group" of the block e, the elements of which would be compatibility classes of "capped" or "strongly capped" Brauer-friendly $\mathcal{O}Ge$-modules. This could be a way to generalise the results of [La-2012] on endo-p-permutation modules. Indeed, we will see that a Brauer-friendly module over the principal block of a finite group is necessarily an endo-p-permutation module, whereas this is no longer true for a nonprincipal block.

The functoriality of the slash construction could make it more comfortable to deal with complexes of modules with endopermutation sources, provided that these modules are Brauer-friendly and compatible with one another. However, the present thesis will only be concerned with single modules, in relation with Morita or stable equivalences between blocks.

Let us now review the organisation of this chapter. Section 3.1 extends Green's theory of vertices and sources to take subpairs into consideration, following [Si-1990]. We study the behaviour of vertex subpairs and sources with respect to the restriction to a local subgroup. Then, in Section 3.2, we rephrase the notion of a fusion-stable source, which appears in [Li-2013], from the language of source algebras and fusion systems to the language of block algebras and Brauer categories. We define the Brauer-friendly $\mathcal{O}Ge$-modules, *i.e.* the direct sums of indecomposable modules that admit compatible fusion-stable endopermutation sources relative to their respective vertex subpairs. We prove that an $\mathcal{O}Ge$-module M is Brauer-friendly if, and only if, the corresponding module iM over the source algebra $A = i\mathcal{O}Gi$ is an endopermutation $\mathcal{O}D$-module.

In section 3.3, we prove that the slash construction can be applied directly to a Brauer-friendly module, and that the result is again a Brauer-friendly module. We check that this construction satisfies some of the properties listed in Fact 3.1, and that it is compatible with the usual slash construction at the level of source algebras.

Section 3.4 deals with the functoriality of the slash construction. We prove that the slash construction can be turned into a functor when it is restricted to a category of compatible endopermutation modules over a given p-group. Then we do the same for a category of compatible Brauer-friendly modules over a given block algebra.

In section 3.5, we complete our survey of the properties of the slash construction by giving a parametrisation of the indecomposable modules in a given Brauer-friendly category, which generalises the one-to-one correspondence of Fact 3.1 (iv). This special case of the Puig correspondence will be an essential tool for the construction of a stable equivalence of Morita type in Chapter 4.

Section 3.6 is more prospective, and contains more questions than we can answer yet. We go a little deeper into the properties of the Brauer functor, in connection with the Brauer category of a block, and we study a categorical structure that looks very much like a representation of a "Brauer 2-category". Then we suggest a possible generalisation of these ideas to slash functors.

3.1 Vertex subpairs and the Green correspondence

Let G be a finite group, and e be a block of the algebra $\mathcal{O}G$. The following lemma extends Green's theory of vertices and sources, as well as the Green correspondence, to take e-subpairs into consideration.

Lemma 3.2. *Let M be an indecomposable $\mathcal{O}Ge$-module and (P, e_P) be an e-subpair of the group G. The following conditions are equivalent.*

(i) The p-subgroup P is contained in a vertex of M, and M is isomorphic to a direct summand of the $\mathcal{O}Ge$-module $e\mathcal{O}Ge_P \otimes_{\mathcal{O}P} V$ for some indecomposable $\mathcal{O}P$-module V.

(ii) The p-subgroup P contains a vertex of M, and the $\mathcal{O}P$-module $e_P M$ admits an indecomposable direct summand V with vertex P.

(iii) The $\mathcal{O}N_G(P, e_P)$-module $e_P M$ admits an indecomposable direct summand L with vertex P such that M is isomorphic to a direct summand of the induced module $\mathrm{Ind}_{N_G(P,e_P)}^{G} L$

(iv) The p-group P is a vertex of M, and the Green correspondent of M with respect to this vertex is an $\mathcal{O}N_G(P)$-module M' that belongs to the block $e'_P = \mathrm{Tr}_{N_G(P,e_P)}^{N_G(P)} e_P$ of the algebra $\mathcal{O}N_G(P)$.

If these conditions are satisfied, then the $\mathcal{O}N_G(P, e_P)$-module L of (iii) and the $\mathcal{O}N_G(P)$-module M' of (iv) satisfy the relations

$$M' \;\simeq\; \mathrm{Ind}_{N_G(P,e_P)}^{N_G(P)} L \quad \textit{and} \quad L \;\simeq\; e_P M'.$$

Proof. We suppose that (iv) is satisfied, and we consider the $\mathcal{O}N_G(P, eP)$-module $L = e_P M'$. Since M' is a direct summand of the restriction $\mathrm{Res}_{N_G(P)}^{G} M$, the $\mathcal{O}N_G(P, e_P)$-module L is a direct summand of $e_P M$. We know from [Ha-2007, Theorem 1.6] that the bimodules $e_P \mathcal{O}N_G(P)$ and $\mathcal{O}N_G(P)e_P$ induce a Morita equivalence $\mathcal{O}N_G(P,e_P)e_P \sim \mathcal{O}N_G(P)e'_P$. The $\mathcal{O}N_G(P)e'_P$-module M' is indecomposable, so the $\mathcal{O}N_G(P,e_P)e_P$-module $L \simeq e_P\mathcal{O}N_G(P) \otimes_{\mathcal{O}N_G(P)e'_P} M'$ is indecomposable, and there is an isomorphism

$$M' \;\simeq\; \mathcal{O}N_G(P)e_P \otimes_{\mathcal{O}N_G(P,e_P)e_P} L \;\simeq\; \mathrm{Ind}_{N_G(P,e_P)}^{N_G(P)} L.$$

Since M is isomorphic to a direct summand of the induced module $\mathrm{Ind}_{N_G(P)}^G M'$, it follows from the transitivity of the induction that M is isomorphic to a direct summand of the induced module $\mathrm{Ind}_{N_G(P,e_P)}^G L$. We have proven that the condition (iv) implies (iii).

We now suppose that (iii) is satisfied. The restriction $\mathrm{Res}_P^{N_G(P,e_P)} L$ admits a direct summand V with vertex P, so the $\mathcal{O}P$-module e_PM also admits V as a direct summand. The induced module $\mathrm{Ind}_{N_G(P,e_P)}^G L$ is relatively P-projective, so the $\mathcal{O}G$-module M is relatively P-projective. Since V is also a direct summand of M and admits P as a vertex, it follows that the p-group P is a vertex of M. Moreover, the $\mathcal{O}P$-module V is a source of the indecomposable $\mathcal{O}N_G(P,e_P)e_P$-module L with respect to the vertex P. Thus L is isomorphic to a direct summand of the $\mathcal{O}N_G(P,e_P)e_P$-module $\mathcal{O}N_G(P,e_P)e_P \otimes_{\mathcal{O}P} V$, and M is isomorphic to a direct summand of the $\mathcal{O}Ge$-module

$$e\,\mathrm{Ind}_{N_G(P,e_P)}^G\Big(\mathcal{O}N_G(P,e_P)e_P \otimes_{\mathcal{O}P} V\Big) \;\simeq\; e\mathcal{O}Ge_P \otimes_{\mathcal{O}P} V.$$

We have proven that the condition in (iii) implies (i) and (ii).

Next, we suppose that (i) is satisfied. The $\mathcal{O}G$-module M is relatively P-projective and the p-group P is contained in a vertex of M, so P itself is a vertex of M. Moreover the $\mathcal{O}G$-module M is isomorphic to a direct summand of the induced module

$$\mathcal{O}Ge_P \otimes_{\mathcal{O}P} V \;\simeq\; \mathrm{Ind}_{N_G(P)}^G\big(\mathcal{O}N_G(P)e_P \otimes_{\mathcal{O}P} V\big),$$

so there exists an indecomposable direct summand M' of the $\mathcal{O}N_G(P)e'_P$-module $\mathcal{O}N_G(P)e_P \otimes_{\mathcal{O}P} V$ such that M is isomorphic to a direct summand of $\mathrm{Ind}_{N_G(P)}^G M'$. The $\mathcal{O}N_G(P)$-module M' is therefore a Green correspondent of M with respect to the vertex P, so that (i) implies (iv).

Finally, we suppose that (ii) is satisfied. Then the restriction $\mathrm{Res}_P^G M$ admits an indecomposable direct summand with vertex P, and the p-group P contains a vertex of M, so P itself is a vertex of M. Let M' be a Green correspondent of M with respect to this vertex, and let f be the block of the algebra $\mathcal{O}N_G(P)$ such that $fM' = M'$. Then the uniqueness of the Green correspondent implies that no indecomposable direct summand of the $\mathcal{O}P$-module $(1-f)M$ admits P as a vertex. Since $e_PM = fe_PM \oplus (1-f)e_PM$, it follows that $fe_P \neq 0$, so that $f = \mathrm{Tr}_{N_G(P,e_P)}^{N_G(P)} e_P$. This proves that (ii) implies (iv), so the four conditions are pairwise equivalent.

The additional statement follows from the uniqueness of the Green correspondent M'. □

We derive from the above lemma a few definitions and easy consequences. Let M be an indecomposable $\mathcal{O}Ge$-module. It follows from Nagao's theorem that there exists an e-subpair (P, e_P) of the the group G that satisfies the statement in Lemma 3.2 (iv), hence the statements in (i), (ii) and (iii). Such an e-subpair is called a vertex subpair of the indecomposable module M (this is consistent with [Si-1990, Definition 2.6]). A source of M with respect to the vertex subpair (P, e_P) is an $\mathcal{O}P$-module V that satisfies any one of the equivalent conditions (i) and (ii). A source triple of M is a triple (P, e_P, V), where (P, e_P) is a vertex subpair of M and V is a source of M with respect to this vertex subpair. The source triples of M form an orbit under the action of the group G by conjugation.

A Green correspondent of M with respect to the vertex subpair (P, e_P) is an $\mathcal{O}N_G(P, e_P)$-module L that satisfies the condition in (iii). This is equivalent to requiring that L be an indecomposable $\mathcal{O}N_G(P, e_P)$-module with vertex P such that the $\mathcal{O}G$-module M is isomorphic to a direct summand of $\mathrm{Ind}_{N_G(P,e_P)}^G L$, or such that L is isomorphic to a direct summand of the $\mathcal{O}N_G(P, e_P)$-module e_PM. The map $M \mapsto L$ induces a one-to-one correspondence between the isomorphism classes of indecomposable $\mathcal{O}Ge$-modules with vertex subpair (P, e_P), and the isomorphism classes of indecomposable $\mathcal{O}N_G(P, e_P)e_P$-modules with vertex P. With these notations, a source of the indecomposable $\mathcal{O}N_G(P, e_P)e_P$-module L with respect to the vertex P is a source of the indecomposable $\mathcal{O}Ge$-module M with respect to the vertex subpair (P, e_P).

We give additionnal characterisations of vertex subpairs and sources in terms of primitive idempotents and source idempotents. These are closer to the original approach of [Si-1990], and will be useful in subsequent proofs.

Lemma 3.3. *Let M be an indecomposable $\mathcal{O}Ge$-module and (P, e_P, V) be a source triple of M.*

(i) There exists a primitive idempotent i of the algebra $(\mathcal{O}G)^P$ such that $\bar{e}_P \operatorname{br}_P(i) \neq 0$ and such that M is isomorphic to a direct summand of the $\mathcal{O}Ge$-module $\mathcal{O}Gi \otimes_{\mathcal{O}P} V$.

(ii) There exists a defect group D of the block e such that $P \leqslant D$, there exists a primitive idempotent j of the algebra $(\mathcal{O}G)^D$ such that $\operatorname{br}_D(j) \neq 0$ and $\bar{e}_P \operatorname{br}_P(j) \neq 0$, and such that the $\mathcal{O}Ge$-module M is isomorphic to a direct summand of the $\mathcal{O}Ge$-module $\mathcal{O}Gj \otimes_{\mathcal{O}D} \operatorname{Ind}_P^D V$.

Proof. If (P, e_P, V) is a source triple of M, then by definition M is a direct summand of the $\mathcal{O}Ge$-module $e\mathcal{O}Ge_P \otimes_{\mathcal{O}P} V$. Consider a decomposition $ee_P = i_1 + \ldots + i_n$ of the idempotent ee_P into mutually orthogonal primitive idempotents in the algebra $(e_P\mathcal{O}Gee_P)^P$. Then the $\mathcal{O}Ge$-module $e\mathcal{O}Ge_P \otimes_{\mathcal{O}P} V$ decomposes as

$$e\mathcal{O}Ge_P \otimes_{\mathcal{O}P} V = (\mathcal{O}Gi_1 \otimes V) \oplus \ldots \oplus (\mathcal{O}Gi_n \otimes V).$$

Since the $\mathcal{O}Ge$-module M is indecomposable, it follows from the Krull-Schmidt theorem that M is isomorphic to a direct summand of the $\mathcal{O}Ge$-module $\mathcal{O}Gi_l \otimes_{\mathcal{O}P} V$ for some integer $l \in \{1, \ldots, n\}$. If $\operatorname{br}_P(i_l) = 0$, then we know from Rosenberg's lemma that the idempotent i_l lies in $\operatorname{Tr}_Q^P((\mathcal{O}G)^Q)$ for some proper subgroup Q of P. Then the $\mathcal{O}(G \times P)$-module $\mathcal{O}Gi_l$ is a direct summand of the induced module $\operatorname{Ind}_{G\times Q}^{G\times P} \mathcal{O}G \simeq \mathcal{O}G \otimes_{\mathcal{O}Q} \mathcal{O}P$, and M is a direct summand of the $\mathcal{O}G$-module $\operatorname{Ind}_Q^G \operatorname{Res}_Q^P V$. Thus M cannot admit P as a vertex. This contradiction proves (i).

Let α be the point of the algebra $(\mathcal{O}Ge)^P$ that contains the idempotent $i = i_l$. Since $\operatorname{br}_P(i) \neq 0$, the pointed group P_α is local. Let D_β be a maximal local pointed group on the algebra $\mathcal{O}Ge$ such that $P_\alpha \leqslant D_\beta$. By [Th-1995], the p-group D is a defect group of the block e and there exists an idempotent $j \in \beta$ such that $ij = ji = i$. Since the pointed group D_β is local, we get $\operatorname{br}_D(j) \neq 0$. Moreover $\bar{e}_P \operatorname{br}_P(j) \operatorname{br}_P(i) = \bar{e}_P \operatorname{br}_P(i) \neq 0$, so we obtain $\bar{e}_P \operatorname{br}_P(j) \neq 0$. Finally, $ji = i$ so the $\mathcal{O}Ge$-module $\mathcal{O}Gi \otimes_{\mathcal{O}P} V$ is a direct summand of

$$\mathcal{O}Gj \otimes_{\mathcal{O}P} V \;\simeq\; \mathcal{O}Gj \otimes_{\mathcal{O}D} \operatorname{Ind}_P^D V.$$

The following result gives some information on the behaviour of vertex subpairs and sources with respect to restriction.

Theorem 3.4. *Let M be an $\mathcal{O}Ge$-module, and H be a subgroup of G. Let (Q, e_Q, W) be a source triple of an indecomposable direct summand of the restriction $\mathrm{Res}^G_H M$. Assume that the subgroup H contains the centraliser $C_G(Q)$. Then (Q, e_Q) is an e-subpair of the group G, and there exists a source triple (P, e_P, V) of an indecomposable direct summand of the $\mathcal{O}Ge$-module M such that $(Q, e_Q) \leqslant (P, e_P)$ and that the $\mathcal{O}Q$-module W is a direct summand of the restriction $\mathrm{Res}^P_Q V$.*

Proof. It is enough to prove the theorem when the $\mathcal{O}Ge$-module M is indecomposable. Thus, by Lemma 3.3, it is enough to consider the following situation. Let (P, e_P) be an e-subpair and V be an $\mathcal{O}P$-module; let $i \in (\mathcal{O}Ge)^P$ be a primitive idempotent such that $\bar{e}_P \,\mathrm{br}_P(i) \neq 0$; let X be an indecomposable direct summand of the $\mathcal{O}H$-module $\mathcal{O}Gi \otimes_{\mathcal{O}P} V$; let (Q, e_Q, W) be a source triple of the indecomposable $\mathcal{O}H$-module X. We assume that the centraliser $C_G(Q)$ is contained in H, and we prove that there exists a G-conjugate (P', e'_P, V') of the triple (P, e_P, V) such that $(Q, e_Q) \leqslant (P', e'_P)$ and that W is a direct summand of $\mathrm{Res}^{P'}_Q V'$.

Since $C_G(Q) \leqslant H$, the block e_Q of $\mathcal{O}C_H(Q)$ is also a block of the algebra $\mathcal{O}QC_G(Q) = \mathcal{O}QC_H(Q)$. Let Y be a Green correspondent of the indecomposable $\mathcal{O}H$-module X with respect to the vertex subpair (Q, e_Q). Then Y is an indecomposable $\mathcal{O}N_H(Q, e_Q)$-module with source triple (Q, e_Q, W). As a consequence, there exists an indecomposable direct summand Z of the restriction $\mathrm{Res}^{N_G(Q,e_Q)}_{QC_G(Q)} Y$ with source triple (Q, e_Q, W). Up to replacing X by Z, we may now suppose that $H = QC_G(Q)$, and that X is an indecomposable direct summand of the $\mathcal{O}H$-module $L = e_Q\mathcal{O}Gi \otimes_{\mathcal{O}P} V$ with vertex Q and source W.

We consider the $\mathcal{O}H$-module $L_0 = \mathcal{O}G \otimes_{\mathcal{O}P} V$. The map $u : L_0 \to L_0$ defined by $u(g \otimes v) = e_Q gi \otimes v$ is an idempotent endomorphism of the $\mathcal{O}H$-module L_0. It satisfies $u(L_0) = L$, so that L is a direct summand of L_0. Let $L_0 = X_0 \oplus \cdots \oplus X_n$ be a Krull-Schmidt decomposition of the $\mathcal{O}H$-module L_0 that refines the decomposition $L_0 = L \oplus \ker u$. We may suppose that $X_0 = X$ and that $L = X_0 \oplus \cdots \oplus X_m$ for some $m \in \{1, \ldots, n\}$.

Let $\mathcal{R}$ be a set of representatives in the group G for the double class set

$H \setminus G/P$. Then we have the decomposition $L_0 = \bigoplus_{g\in\mathcal{R}} L_g$, where

$$L_g = \mathcal{O}HgP \otimes_{\mathcal{O}P} V$$

for any element $g \in \mathcal{R}$. Let $L_0 = Y_0 \oplus \cdots \oplus Y_n$ be a Krull-Schmidt decomposition of the $\mathcal{O}H$-module L_0 that refines the decomposition $L_0 = \bigoplus_{g\in\mathcal{R}} L_g$.

From the proof of the Krull-Schmidt theorem in [Be-1991], we know that there exists an integer $l \in \{1, \ldots, n\}$ such that the projection of Y_l on X_0 along $X_1 \oplus \cdots \oplus X_n$ is an isomorphism. We may suppose $l = 0$. Then $u(Y_0)$ is a complement of the direct summand $X_1 \oplus \cdots \oplus X_m$ in the $\mathcal{O}H$-module L. In particular, the $\mathcal{O}H$-modules $u(Y_0)$ and X_0 are isomorphic, so we may suppose $X_0 = u(Y_0)$.

By construction, there exists an element $g \in \mathcal{R}$ such that the $\mathcal{O}H$-module Y_0 is a direct summand of L_g. We have an isomorphism

$$L_g \;\simeq\; \mathrm{Ind}_{H\cap {}^gP}^{H} \mathrm{Res}_{H\cap {}^gP}^{{}^gP}\, gV,$$

so the $\mathcal{O}H$-module L_g, and its direct summand Y_0, are relatively $H \cap {}^gP$-projective. Since the vertex Q of the $\mathcal{O}H$-module $Y_0 \simeq X_0$ is normal in H, it follows that $Q \leqslant {}^gP$. Assume for a moment that $(Q, e_Q) \not\leqslant {}^g(P, e_P)$. For any $h \in H$, we have

$$u(hgP \otimes_{\mathcal{O}P} V) \;=\; e_Q(hgP)i \otimes_{\mathcal{O}P} V \;=\; e_Q({}^{hg}i).(hgP) \otimes_{\mathcal{O}P} V.$$

The e-subpair (Q, e_Q) is normalised by the group H, so we have $Q \leqslant {}^{hg}P$ but $(Q, e_Q) \not\leqslant {}^{hg}(P, e_P)$. By definition, i is a primitive idempotent of $(\mathcal{O}Ge)^P$ such that $\bar{e}_P\, \mathrm{br}_P(i) \neq 0$. Thus ${}^{hg}i$ is a primitive idempotent in $(\mathcal{O}Ge)^Q$, and it follows from [Th-1995, Lemma 40.1] that $\bar{e}_Q\, \mathrm{br}_Q({}^{hg}i) = 0$. Then we obtain

$$u(hgP \otimes V) \;\subseteq\; J_Q\, L_0,$$

where $J_Q \subseteq (\mathcal{O}Ge)^Q$ is the kernel of the Brauer map br_Q. This implies

$$X_0 \;=\; u(Y_0) \;\subseteq\; u(L_g) \;\subseteq\; J_Q\, L_0.$$

We know that X_0 is a direct summand of the $\mathcal{O}H$-module L_0. Consider the endomorphism algebra $A = \mathrm{End}_{\mathcal{O}}(L_0)$, and let $v \in A^H$ be an idempotent such that $X_0 = vL_0$. Then there is an isomorphism $\mathrm{End}_{\mathcal{O}}(X_0) \simeq vAv$, which induces an isomorphism $\mathrm{Br}_Q(\mathrm{End}_{\mathcal{O}}(X_0)) \simeq \mathrm{br}_Q^A(vAv)$. The inclusion $X_0 \subseteq J_Q L_0$ yields $vAv \subseteq J_Q A$, hence $\mathrm{br}_Q^A(vAv) = 0$. We obtain $\mathrm{Br}_Q(\mathrm{End}_{\mathcal{O}}(X_0)) = 0$, a contradiction since Q is a vertex of X_0.

This contradiction proves that $(Q, e_Q) \leqslant {}^g(P, e_P)$. Then the $\mathcal{O}H$-module $X = X_0$ is isomorphic to a direct summand of the induced module

$$L_g \;\simeq\; \mathrm{Ind}_{H\cap {}^gP}^{H} \mathrm{Res}_{H\cap {}^gP}^{{}^gP}\, gV.$$

As a consequence, the source W of X is isomorphic to a direct summand of the $\mathcal{O}Q$-module $\mathrm{Res}_Q^{{}^{hg}P}\, hgV$ for some $h \in H$. Since $H = QC_G(Q)$, we have $\mathrm{Res}_Q^{{}^{hg}P}\, hgV \simeq \mathrm{Res}_Q^{{}^gP}\, gV$. Then we set $(P', e'_P, V') = {}^g(P, e_P, V)$, and the proof is complete. □

The typical application of Theorem 3.4 that we have in mind is as follows. Let M be an indecomposable $\mathcal{O}Ge$-module, and let (Q, e_Q) be an e-subpair of the group G. Let H be a subgroup of G such that $QC_G(Q) \leqslant H \leqslant N_G(Q, e_Q)$. Let (R, e_R, W) be a source triple of an indecomposable direct summand of the $\mathcal{O}H$-module $e_Q M$. If the vertex R contains the p-group Q, then we have $C_G(R) \leqslant C_G(Q) \leqslant H$. Thus the theorem applies: there exists a source triple (P, e_P, V) of M such that $(R, e_R) \leqslant (P, e_P)$ and that the $\mathcal{O}R$-module W is a direct summand of the restriction $\mathrm{Res}_R^P V$.

This statement is unlikely to be true without the assumption $Q \leqslant R$. However, we might try to replace the restriction $e_Q \mathrm{Res}_H^G M$ with the restriction through a smaller $(\mathcal{O}He_Q, \mathcal{O}Ge)$-bimodule. The following lemma is a straightforward generalisation of [ALR-2001, Theorem 5 (i)].

Lemma 3.5. *Let (Q, e_Q) be an e-subpair of the group G, and H be a subgroup of G such that $QC_G(Q) \leqslant H \leqslant N_G(Q, e_Q)$. Up to isomorphism, the $\mathcal{O}(H \times G)$-module $e_Q\mathcal{O}Ge$ admits a unique indecomposable direct summand X with vertex ΔR, where R is a defect group of the block e_Q in the group $N_G(Q, e_Q)$.*

Proof. Let R be a defect group of the block e_Q in the group H, and let ΔR be the direct image of the p-group R by the diagonal embedding $h \mapsto (h, h)$ of the group H into the direct product $H \times G$. The $\mathcal{O}(H \times H)$-module $e_Q\mathcal{O}H$ is indecomposable with vertex ΔR. Moreover, the normaliser $N_{H\times G}(\Delta R)$ is the subgroup $(C_H(R) \times C_G(R))\Delta N_H(R)$, which is contained in $H \times H$ since the defect group R contains the normal p-subgroup Q of H. Then it follows from the Green correspondence that the induced module $\mathrm{Ind}_{H\times H}^{H\times G} e_Q\mathcal{O}H \simeq e_Q\mathcal{O}G$ admits a unique direct summand X with vertex ΔR. Moreover the $\mathcal{O}(H \times H)$-module $e_Q\mathcal{O}H$ belongs to the block $e_Q \otimes e_Q^{\mathrm{o}}$, so we deduce from Nagao's theorem that the Green correspondent X belongs to the block $e_Q \otimes e^{\mathrm{o}}$ of the algebra $\mathcal{O}(H \times G)$, *i.e.*, that X is actually a direct summand of the $\mathcal{O}(H \times G)$-module $e_Q\mathcal{O}Ge$.

If R' is another defect group of the block e_Q, then R' is H-conjugate to the defect group R, and the diagonal subgroup $\Delta R'$ is $(H \times G)$-conjugate to the vertex ΔR. Hence $\Delta R'$ is another vertex of the $\mathcal{O}(H \times G)$-module X. Thus the definition of X does not depend on the choice of the defect group R. □

Following [Rq-0000], we say that the $(\mathcal{O}He_Q, \mathcal{O}Ge)$-bimodule X is the Green bimodule attached to the e-subpair (Q, e_Q) and the local subgroup H of G. We are led to ask the following question. We will not try to answer it here, since we are eventually interested in the slashed module $M(Q, e_Q)$, which only depends on the direct summands of $e_Q \mathrm{Res}_H^G M$ with a vertex that contains Q.

Question 3.6. *Let (Q, e_Q) be an e-subpair of the group G, and H be a subgroup of G such that $QC_G(Q) \leqslant H \leqslant N_G(Q, e_Q)$. Let X be the corresponding Green bimodule. Let M be an indecomposable $\mathcal{O}Ge$-module and let (R, e_R, W) be a source triple of an indecomposable direct summand of the $\mathcal{O}H$-module $X \otimes_{\mathcal{O}G} M$. Does there always exist a source triple (P, e_P, V) of M such that $(R, e_R) \leqslant (P, e_P)$ (in a sense that would need to be clarified) and such that the $\mathcal{O}R$-module W is a direct summand of the restriction $\mathrm{Res}_R^P V$?*

3.2 Brauer-friendly modules

In this section, we define and study a class of $\mathcal{O}Ge$-modules with endopermutation sources, where G is a finite group and e is a block of the algebra $\mathcal{O}G$. The following definition is derived from [Li-2013].

Definition 3.7. Let (P, e_P) be an e-subpair of the group G, and let V be an endopermutation $\mathcal{O}P$-module. We say that V is fusion-stable in the group G with respect to the subpair (P, e_P) if the endopermutation $\mathcal{O}Q$-modules

$\mathrm{Res}_{\phi_1} V$ and $\mathrm{Res}_{\phi_2} V$ are compatible for any e-subpair (Q, e_Q) and any two arrows $\phi_1, \phi_2 : (Q, e_Q) \to (P, e_P)$ in the Brauer category $\mathbf{Br}(G, e)$. To make it shorter, we say that (P, e_P, V) is a fusion-stable endopermutation source triple in the group G if V is an indecomposable capped endopermutation $\mathcal{O}P$-module that is fusion-stable in G with respect to the subpair (P, e_P).
We say that two fusion-stable endopermutation source triples (P_1, e_1, V_1) and (P_2, e_2, V_2) are compatible if the endopermutation $\mathcal{O}Q$-modules $\mathrm{Res}_{\phi_1} V_1$ and $\mathrm{Res}_{\phi_2} V_2$ are compatible for any e-subpair (Q, e_Q) and any two arrows $\phi_1 : (Q, e_Q) \to (P_1, e_1)$, $\phi_2 : (Q, e_Q) \to (P_2, e_2)$ in the Brauer category $\mathbf{Br}(G, e)$.

In practice, if (P, e_P) is an e-subpair and V an endopermutation $\mathcal{O}P$-module, checking that the endopermutation $\mathcal{O}Q$-modules $\mathrm{Res}^P_Q V$ and $\mathrm{Res}^{gP}_Q gV$ are compatible, for any e-subpair (Q, e_Q) contained in (P, e_P) and any element $g \in G$ such that $(Q, e_Q) \leqslant {}^g(P, e_P)$, is enough to conclude that (P, e_P, V) is a fusion-stable endopermutation source triple.

Let us first consider a classical case. We assume that e is a nilpotent block of the group G, as defined in [BP-1980]. We let D be a defect group of the block e. By assumption, the subgroup D controls the e-fusion in the group G; in particular, any endopermutation $\mathcal{O}D$-module is fusion-stable in G with respect to any subpair of the form (D, e_D). Let (P, e_P) be an e-subpair of the group G such that $P \leqslant D$, and let V be a capped indecomposable $\mathcal{O}P$-module. Then it follows from Dade's characterisation of endopermutation induced modules in [Da-1978, Proposition 2.15] that (P, e_P, V) is a fusion-stable endopermutation source triple if, and only if, the induced module $\mathrm{Ind}^D_P V$ is an endopermutation $\mathcal{O}D$-module. The latter condition is always satisfied if the defect group D is abelian, or if the defect group D is a direct product $R \times R$ and the vertex P is the diagonal subgroup ΔR (this is the standard setting in the context of basic Morita equivalences, as defined in [Pu-1999]).

Things are more complicated for an arbitrary block. The following lemma gives a few rules to deal with fusion-stable sources.

Lemma 3.8. *With the above notations, let (P_1, e_1, V_1) and (P_2, e_2, V_2) be two source triples in the group G with respect to the block e.*

(i) If (P_1, e_1, V_1) is a fusion-stable endopermutation source triple, then any G-conjugate of (P_1, e_1, V_1) is a fusion-stable endopermutation source triple. If

moreover (P_2, e_2, V_2) *is a fusion-stable endopermutation source triple that is compatible with* (P_1, e_1, V_1)*, then it is also compatible with any* G*-conjugate of* (P_1, e_1, V_1)*.*

(ii) Let (R, e_R) *be an* e*-subpair and* $\psi_1 : (P_1, e_1) \to (R, e_R)$*,* $\psi_2 : (P_2, e_2) \to (R, e_R)$ *be arrows in the Brauer category* $\mathbf{Br}(G, e)$*. Then* (P_1, e_1, V_1) *and* (P_2, e_2, V_2) *are compatible fusion-stable endopermutation source triples if, and only if, the direct sum* $\operatorname{Ind}_{\psi_1} V_1 \oplus \operatorname{Ind}_{\psi_2} V_2$ *is an endopermutation* $\mathcal{O}R$*-module that is fusion-stable in the group* G *with respect to the subpair* (R, e_R)*.*

(iii) For $i = 1, 2$*, let* (Q_i, f_i) *be a subpair of* (P_i, e_i)*, and let* W_i *be a capped indecomposable direct summand of the restriction* $\operatorname{Res}^{P_i}_{Q_i} V_i$*. If* (P_1, e_1, V_1) *and* (P_2, e_2, V_2) *are compatible fusion-stable endopermutation source triples, then* (Q_1, f_1, W_1) *and* (Q_2, f_2, W_2) *are compatible fusion-stable endopermutation source triples.*

Proof. Let g be an element of the group G, and set $(P_1', e_1', V_1') = {}^g(P_1, e_1, V_1)$. Let $\alpha : (P_1', e_1') \to (P_1, e_1)$ be the arrow in the Brauer category $\mathbf{Br}(G, e)$ that is induced by the element g^{-1}, so that $V_1' = \operatorname{Res}_\alpha V_1$. Suppose that (P_1, e_1, V_1) is a fusion-stable endopermutation source triple. For any e-subpair (Q, e_Q) and any two arrows $\phi_1', \phi_2' : (Q, e_Q) \to (P_1', e_1')$ in the Brauer category $\mathbf{Br}(G, e)$, we set $\phi_1 = \alpha \circ \phi_1'$ and $\phi_2 = \alpha \circ \phi_2'$. Then $\phi_1, \phi_2 : (Q, e_Q) \to (P_1, e_1)$ are two arrows in the Brauer category $\mathbf{Br}(G, e)$. It follows that the endopermutation $\mathcal{O}Q$-modules $\operatorname{Res}_{\phi_1'} V_1' = \operatorname{Res}_{\phi_1} V_1$ and $\operatorname{Res}_{\phi_2'} V_1' = \operatorname{Res}_{\phi_2} V_1$ are compatible. So (P_1', e_1', V_1') is a fusion-stable endopermutation source triple. If the source triples (P_1, e_1, V_1) and (P_2, e_2, V_2) are compatible, we prove similarly that the source triples (P_1', e_1', V_1') and (P_2, e_2, V_2) are compatible. This proves (i).

Let (R, e_R) be an e-subpair and $\psi_1 : (P_1, e_1) \to (R, e_R)$, $\psi_2 : (P_2, e_2) \to (R, e_R)$ be arrows in the Brauer category $\mathbf{Br}(G, e)$. Suppose that (P_1, e_1, V_1) and (P_2, e_2, V_2) are compatible fusion-stable endopermutation source triples. Up to replacing these triples by conjugates, we may suppose that the e-subpairs (P_1, e_1) and (P_2, e_2) are contained in (R, e_R), and that the arrows ψ_1 and ψ_2 are the inclusion maps. We follow the lines of the proof of [Da-1978, Lemma 6.8]. The Mackey formula gives an isomorphism of $\mathcal{O}\Delta R$-modules

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}}(\mathrm{Ind}_{P_1}^{R} V_1, \mathrm{Ind}_{P_2}^{R} V_2) &\simeq \mathrm{Res}_{\Delta R}^{R\times R} \mathrm{Ind}_{P_2\times P_1}^{R\times R} \mathrm{Hom}_{\mathcal{O}}(V_1, V_2)\\ &\simeq \bigoplus_{x\in P_2\backslash R/P_1} \mathrm{Ind}_{\Delta(P_2\cap {}^xP_1)}^{\Delta R} \mathrm{Res}_{\Delta(P_2\cap {}^xP_1)}^{P_2\times {}^xP_1} \mathrm{Hom}_{\mathcal{O}}(xV_1, V_2)\\ &\simeq \bigoplus_{x\in P_2\backslash R/P_1} \mathrm{Ind}_{\Delta Q^{(x)}}^{\Delta R} \mathrm{Hom}_{\mathcal{O}}(\mathrm{Res}_{\phi_1^{(x)}} V_1, \mathrm{Res}_{\phi_1^{(x)}} V_2),\end{aligned}$$

where $Q^{(x)} = P_2 \cap {}^xP_1$, the morphism $\phi_1^{(x)} : Q^{(x)} \to P_1$ sends an element $y \in Q^{(x)}$ to ${}^{x^{-1}}y \in P_1$, and the morphism $\phi_2^{(x)} : Q^{(x)} \to P_2$ is the inclusion map. Let $e_Q^{(x)}$ be the block of $\mathcal{O}C_G(Q^{(x)})$ such that $(Q^{(x)}, e_Q^{(x)}) \leqslant (R, e_R)$. Since the element x lies in the p-group R, we also have the inclusion ${}^{x^{-1}}(Q^{(x)}, e_Q^{(x)}) \leqslant (R, e_R)$. By assumption, the subpair (P_2, e_2) is contained in (R, e_R), so the inclusion $Q^{(x)} \leqslant P_2$ implies $(Q^{(x)}, e_Q^{(x)}) \leqslant (P_2, e_2)$. Similarly, the subpairs (P_1, e_1) is contained in (R, e_R), so the inclusion ${}^{x^{-1}}Q^{(x)} \leqslant P_2$ implies ${}^{x^{-1}}(Q^{(x)}, e_Q^{(x)}) \leqslant (P_2, e_2)$. Thus $\phi_1^{(x)} : (Q^{(x)}, e_Q^{(x)}) \to (P_1, e_1)$ and $\phi_2^{(x)} : (Q^{(x)}, e_Q^{(x)}) \to (P_2, e_2)$ are arrows in the Brauer category $\mathbf{Br}(G, e)$. Since the triples (P_1, e_1, V_1) and (P_2, e_2, V_2) are compatible, we deduce that $\mathrm{Hom}_{\mathcal{O}}(\mathrm{Ind}_{P_1}^{R} V_1, \mathrm{Ind}_{P_2}^{R} V_2)$ is a permutation $\mathcal{O}\Delta R$-module.

Similarly, the endomorphism algebras $\mathrm{End}_{\mathcal{O}}(\mathrm{Ind}_{P_1}^{R} V_1)$ and $\mathrm{End}_{\mathcal{O}}(\mathrm{Ind}_{P_2}^{R} V_2)$ are permutation $\mathcal{O}\Delta R$-module. Thus $\mathrm{Ind}_{P_1}^{R} V_1$ and $\mathrm{Ind}_{P_2}^{R} V_2$ are compatible endopermutation $\mathcal{O}R$-modules.

More generally, let (Q, e_Q) be an e-subpair that is contained in (R, e_R) and let $g \in G$ be an element such that $(Q, e_Q) \leqslant {}^g(R, e_R)$. The Mackey formula gives an isomorphism of $\mathcal{O}\Delta Q$-modules

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}}(\mathrm{Res}_{Q}^{R} \mathrm{Ind}_{P_1}^{R} V_1, \mathrm{Res}_{Q}^{gR} g\,\mathrm{Ind}_{P_2}^{R} V_2) &\simeq \mathrm{Res}_{\Delta Q}^{{}^gR\times R} \mathrm{Ind}_{{}^gP_2\times P_1}^{{}^gR\times R} \mathrm{Hom}_{\mathcal{O}}(V_1, gV_2)\\ &\simeq \bigoplus_{(y,x)\in \Delta Q\backslash {}^gR\times R/{}^gP_2\times P_1} \mathrm{Ind}_{\Delta Q^{(y,x)}}^{\Delta Q} \mathrm{Hom}_{\mathcal{O}}(\mathrm{Res}_{\phi_1^{(y,x)}} V_1, \mathrm{Res}_{\phi_2^{(y,x)}} V_2),\end{aligned}$$

where $Q^{(y,x)} = Q \cap {}^{yg}P_2 \cap {}^xP_1$, the map $\phi_1^{(y,x)} : Q^{(y,x)} \to P_1$ sends an element z to ${}^{x^{-1}}z$, and the map $\phi_2^{(y,x)} : Q^{(y,x)} \to P_2$ sends an element z to ${}^{g^{-1}y^{-1}}z$. As above, we deduce that $\mathrm{Hom}_{\mathcal{O}}(\mathrm{Res}_{Q}^{R} \mathrm{Ind}_{P_1}^{R} V_1, \mathrm{Res}_{Q}^{gR} g\,\mathrm{Ind}_{P_2}^{R} V_2)$ is a permutation $\mathcal{O}\Delta Q$-module. It follows that the restrictions

$$\mathrm{Res}_{Q}^{R} \mathrm{Ind}_{P_1}^{R} V_1 \quad ; \quad \mathrm{Res}_{Q}^{R} \mathrm{Ind}_{P_2}^{R} V_2 \quad ; \quad \mathrm{Res}_{Q}^{gR} g\,\mathrm{Ind}_{P_1}^{R} V_1 \quad ; \quad \mathrm{Res}_{Q}^{gR} g\,\mathrm{Ind}_{P_2}^{R} V_2$$

are compatible endopermutation $\mathcal{O}Q$-modules. So the direct sum $\mathrm{Ind}_{P_1}^{R} V_1 \oplus \mathrm{Ind}_{P_2}^{R} V_2$ is an endopermutation $\mathcal{O}R$-module that is fusion-stable with respect to the subpair (R, e_R). This proves one half of (ii). The proof of the converse statement in straightforward. The statement in (iii) can be deduced from (ii) and [Da-1978, Lemma 6.8], or proven directly along the same lines as (i). □

We know from Lemma 3.8 (i) that the notion of compatible fusion-stable endopermutation source triples is invariant by conjugation in the group G. Since the source triples of an indecomposable $\mathcal{O}Ge$-module are defined up to conjugation in G, the following definition is unambiguous.

Definition 3.9. An $\mathcal{O}Ge$-module M is Brauer-friendly if it is a direct sum of indecomposable $\mathcal{O}Ge$-modules with compatible fusion-stable endopermutation source triples. A subcategory ${}_{\mathcal{O}Ge}\mathbf{M}$ of the category ${}_{\mathcal{O}Ge}\mathbf{Mod}$ is Brauer-friendly if ${}_{\mathcal{O}Ge}\mathbf{M}$ is closed under (finite) direct sums and every object in ${}_{\mathcal{O}Ge}\mathbf{M}$ is a Brauer-friendly $\mathcal{O}Ge$-module.

It is clear that the triples $(P, e_P, \mathcal{O})$, where (P, e_P) runs into the set of e-subpairs of the group G and $\mathcal{O}$ is the trivial $\mathcal{O}P$-module, are compatible fusion-stable endopermutation source triples. Thus any p-permutation $\mathcal{O}Ge$-module is Brauer-friendly, and the category ${}_{\mathcal{O}Ge}\mathbf{Perm}$ of p-permutation $\mathcal{O}Ge$-module is a Brauer-friendly subcategory of ${}_{\mathcal{O}Ge}\mathbf{Mod}$.

More generally, let M be an endo-p-permutation $\mathcal{O}Ge$-module, as defined in [Ur-2006, Définition 2.2]. Let (P, e_P, V) be a source triple of an indecomposable direct summand of M. Then, by [Ur-2006, Théorème 2.11], the source V is an endopermutation $\mathcal{O}P$-module and the restrictions $\mathrm{Res}_Q^P V$ and $\mathrm{Res}_Q^{{}^gP} gV$ are compatible endopermutation $\mathcal{O}Q$-modules for any subgroup Q of P and any element $g \in G$ such that $Q \leqslant {}^gP$. This condition is only concerned with p-subgroups of G, and not with e-subpairs. *A fortiori*, the weaker condition of Definition 3.7 is satisfied, so (P, e_P, V) is a fusion-stable endopermutation source triple and M is a Brauer-friendly $\mathcal{O}Ge$-module. However, the category of endo-p-permutation $\mathcal{O}Ge$-modules is not a Brauer-friendly subcategory of ${}_{\mathcal{O}Ge}\mathbf{Mod}$, because it is not closed under direct sums (unless e is a block with defect zero).

Conversely, let M be a Brauer-friendly $\mathcal{O}Ge$-module. If the block e is the principal block of the group G, then an e-subpair (P, e_P) is uniquely determined by the p-subgroup P of G. Thus we can deduce from [Ur-2006, Théorème 2.11] that M is an endo-p-permutation $\mathcal{O}Ge$-module. However, this does not hold for an arbitrary block.

Let us give a more general example.

Lemma 3.10. *Let (P, e_P) be an e-subpair, and let V be an endopermutation $\mathcal{O}P$-module that is fusion-stable with respect to the subpair (P, e_P). Let $i \in (\mathcal{O}Ge)^P$ be an idempotent such that $\bar{e}_P \operatorname{br}_P(i) \neq 0$. Assume that, for any subgroup Q of P, the idempotent $\operatorname{br}_Q(i)$ lies in a single block of the algebra $kC_G(Q)$. Then the $\mathcal{O}Ge$-module $L = \mathcal{O}Gi \otimes_{\mathcal{O}P} V$ is Brauer-friendly.*

Proof. Let X be an indecomposable direct summand of the $\mathcal{O}Ge$-module L, with source triple (Q, e_Q, W). It appears from the proof of Theorem 3.4 that there exists a G-conjugate (P', e'_P, V') of the triple (P, e_P, V) such that $(Q, e_Q) \leqslant (P', e'_P)$ and W is a direct summand of the restriction $\operatorname{Res}^{P'}_Q V'$. Then we deduce from Lemma 3.8 (i) and (iii) that L is a Brauer-friendly $\mathcal{O}Ge$-module. □

Notice that the compatibility of endopermutation modules is preserved by the reduction from the local ring $\mathcal{O}$ to the residue field k. For any Brauer-friendly $\mathcal{O}Ge$-module M, it follows that the reduction $k \otimes_{\mathcal{O}} M$ is a Brauer-friendly kGe-module. The notion is also partially compatible with the restriction to a local subgroup, as appears in the following lemma.

Lemma 3.11. *Let M be a Brauer-friendly $\mathcal{O}Ge$-module. Let (Q, e_Q) be an e-subpair of the group G, and H be a subgroup of G such that $QC_G(Q) \leqslant H \leqslant N_G(Q, e_Q)$.*

(i) The $\mathcal{O}He_Q$-module e_QM admits the decomposition $e_QM = L \oplus L'$, where L is a Brauer-friendly $\mathcal{O}He_Q$-module and L' is a direct sum of indecomposable $\mathcal{O}He_Q$-modules with vertices that do not contain the normal p-group Q.

(ii) The restriction $\operatorname{Res}^H_Q L$ is an endopermutation $\mathcal{O}Q$-module.

Proof. The $\mathcal{O}H$-module e_QM certainly admits the decomposition $e_QM = L \oplus L'$, where L is a direct sum of indecomposable $\mathcal{O}H$-modules with vertices that contain Q and L' is a direct sum of indecomposable $\mathcal{O}H$-modules with vertices that do not contain Q. Let (R, e_R, W) be a source triple of an indecomposable direct summand of the $\mathcal{O}H$-module L. By Theorem 3.4, (R, e_R) is an e-subpair of the group G and there exists a source triple (P, e_P, V) of the $\mathcal{O}Ge$-module M such that $(R, e_R) \leqslant (P, e_P)$ and W is isomorphic to a direct summand of $\mathrm{Res}^P_R V$. In particular, (R, e_R, W) may be seen as a source triple of the group G with respect to the block e. By Lemma 3.8 (iii), the indecomposable direct summands of L have endopermutation source triples that are fusion-stable and compatible in the Brauer category $\mathbf{Br}(G, e)$. *A fortiori*, they are fusion-stable and compatible in the Brauer category $\mathbf{Br}(H, e_Q)$. Thus L is a Brauer-friendly $\mathcal{O}He_Q$-module, and (i) is proven.

Let X, X' be indecomposable direct summands of the $\mathcal{O}H$-module L, and let (R, f, W), (R', f', W') be respective source triples of these indecomposable modules. We know that these are compatible fusion-stable endopermutation source triples. Moreover we have $(Q, e_Q) \leqslant {}^h(R, f)$ and $(Q, e_Q) \leqslant {}^{h'}(R', f')$ for any two elements h, h' of the group H. So the restrictions $\mathrm{Res}^{{}^hR}_Q {}^hW$ and $\mathrm{Res}^{{}^{h'}R'}_Q {}^{h'}W'$ are compatible endopermutation $\mathcal{O}Q$-modules. Then it follows from the Mackey formula that the restrictions $\mathrm{Res}^H_Q \mathrm{Ind}^H_R W$ and $\mathrm{Res}^H_Q \mathrm{Ind}^H_{R'} W'$ are compatible endopermutation $\mathcal{O}Q$-modules. Since the restriction $\mathrm{Res}^H_Q L$ is a direct sum of direct summands of modules of this kind, it is an endopermutation $\mathcal{O}Q$-module. This proves (ii). □

Let M be a Brauer-friendly $\mathcal{O}Ge$-module, and (Q, e_Q) be an e-subpair of the group G. It follows from Lemma 3.11 that there is a unique isomorphism class of capped indecomposable direct summand of the $\mathcal{O}Q$-module e_QM. In particular, if the $\mathcal{O}Ge$-module M is indecomposable and (Q, e_Q) is a vertex subpair of M, this means that the source of M with respect to this vertex subpair is unique up to isomorphism.

We conclude this section with a lemma that connects our notion of Brauer-friendly module with Linckelmann's notion of module with fusion-stable endopermutation source, which appears in [Li-2013, Section 3].

Lemma 3.12. *Let G be a finite group and e be a block of the group algebra $\mathcal{O}G$. Let D be a defect group of the block e and i be a primitive idempotent of the algebra $(\mathcal{O}Ge)^D$ such that $\mathrm{br}_D(i) \neq 0$,* i.e., *a source idempotent of the block e. An $\mathcal{O}Ge$-module M is Brauer-friendly if, and only if, its direct summand iM is an endopermutation $\mathcal{O}D$-module.*

Proof. Let e_D be the unique block of the algebra $\mathcal{O}C_G(D)$ such that $\bar{e}_D\,\mathrm{br}_D(i) \neq$

0. Let $A = i\mathcal{O}i$ be the source algebra attached to the source idempotent i. Let M be an $\mathcal{O}Ge$-module, and let $M = M_1 \oplus \cdots \oplus M_n$ be a Krull-Schmidt decomposition of M. For each integer $l \in \{1, \ldots, n\}$, let (P_l, e_l, V_l) be a source triple of the indecomposable $\mathcal{O}Ge$-module M_l. By Lemma 3.3 (ii), there exists a defect group D_l and a source idempotent j_l of the block e with respect to the defect group D_l such that $P_l \leqslant D_l$ and $\bar{e}_l \operatorname{br}_{P_l}(j_l) \neq 0$, and such that M_l is isomorphic to a direct summand of the $\mathcal{O}Ge$-module $\mathcal{O}Gj_l \otimes_{\mathcal{O}D_l} \operatorname{Ind}_{P_l}^{D_l} V_l$.

We know from [AB-1979] that any maximal e-subpairs of the group G is a conjugate of (D, e_D). Up to replacing the source triple (P_l, e_l, V_l) by a G-conjugate, we may therefore suppose that $D_l = D$ and $\bar{e}_D \operatorname{br}_D(j_l) \neq 0$. We know from [Th-1995, Proposition 40.13] that there is a unique conjugacy class in the algebra $(\mathcal{O}Ge)^D$ of primitive idempotents j such that $\bar{e}_D \operatorname{br}_D(j) \neq 0$. Thus there exists an invertible element $u \in (\mathcal{O}Ge)^D$ such that $j_l = {}^u i$, and there is an isomorphism $\mathcal{O}Gj_l \otimes_{\mathcal{O}D} \operatorname{Ind}_{P_l}^{D} V_l \simeq \mathcal{O}Gi \otimes_{\mathcal{O}D} \operatorname{Ind}_{P_l}^{D} V_l$. If we set $W = \operatorname{Ind}_{P_1}^{D} V_1 \oplus \cdots \oplus \operatorname{Ind}_{P_n}^{D} V_n$, it follows that M is isomorphic to a direct summand of the $\mathcal{O}G$-module $L = \mathcal{O}Gi \otimes_{\mathcal{O}D} W$, and that iM is isomorphic to a direct summand of the A-module $iL = A \otimes_{\mathcal{O}D} W$.

Suppose that the $\mathcal{O}Ge$-module M is Brauer-friendly. Then, by Lemma 3.8 (ii), the $\mathcal{O}D$-module W is an endopermutation $\mathcal{O}D$-module that is fusion-stable with respect to the subpair (D, e_D). By [Li-2013, Proposition 3.2 (i)], this implies that the A-module iL is an endopermutation $\mathcal{O}D$-module. It follows that iM is an endopermutation $\mathcal{O}D$-module.

Conversely, suppose that iM is an endopermutation $\mathcal{O}D$-module. We know from [Th-1995, Section 47] or [Li-2013, Section 2] that the action of the group G on the lattice of subpairs of (D, e_D) can be read in the D-interior algebra A (*cf.* Section 1.5 of the present thesis). Since iM is an A-module, it follows that the endopermutation $\mathcal{O}D$-module iM is fusion-stable in the group G with respect to the subpair (D, e_D). For any integer $l \in \{1, \ldots, n\}$, the source V_l is isomorphic to a capped indecomposable direct summand of the restriction $\operatorname{Res}_{P_l}^{D} iM$. Thus, by Lemma 3.8 (iii), the (P_l, e_l, V_l), $1 \leqslant l \leqslant n$, are compatible fusion-stable endopermutation source triples. So M is a Brauer-friendly $\mathcal{O}Ge$-module. $\square$

3.3 The slash construction

In this section, we prove that Dade's slash construction can be applied to a Brauer-friendly module, and that the result is again a Brauer-friendly module. We begin with a couple of lemmas on the relations between the slash construction and the induction of endopermutation modules.

Lemma 3.13. *Let G be a finite group and Q be a normal p-subgroup of G. Let P be a p-subgroup of G that contains Q and V be an $\mathcal{O}P$-module. Suppose that the restriction $\mathrm{Res}_Q^G \mathrm{Ind}_P^G V$ is an endopermutation $\mathcal{O}Q$-module. Then the $C_G(Q)$-interior G-algebra $\mathrm{Br}_Q(\mathrm{Ind}_P^G \mathrm{End}_{\mathcal{O}}(V))$ admits a (non-unique) extension to a G-interior algebra. Once this extension has been chosen, there is a natural isomorphism of G-interior algebras*

$$\Phi : \mathrm{Ind}_P^G \mathrm{Br}_Q(\mathrm{End}_k(V)) \;\to\; \mathrm{Br}_Q(\mathrm{Ind}_P^G \mathrm{End}_{\mathcal{O}}(V)).$$

Proof. The p-subgroup Q is normal in the group G, hence in the p-group P. Let us write $S = \mathrm{End}_{\mathcal{O}}(V)$, a P-interior matrix algebra over the ring $\mathcal{O}$ which can also be considered as a $\mathcal{O}(P \times P)$-module. The conclusion of the lemma is trivial if the p-subgroup Q is not contained in a vertex of an indecomposable direct summand of the $\mathcal{O}P$-module V, so we will assume that $\mathrm{Br}_Q(S) \neq 0$.

On the one hand, we consider the G-interior algebra $A = \mathrm{Ind}_P^G S$. As a $\mathcal{O}(G \times G)$-module, we have

$$A \;=\; \mathcal{O}G \otimes_{\mathcal{O}P} S \otimes_{\mathcal{O}P} \mathcal{O}G.$$

The map $\phi : S \to A, s \mapsto 1_G \otimes s \otimes 1_G$ is an embedding of algebras. It induces an isomorphism of P-algebras $S \simeq \alpha A \alpha$, where $\alpha = \phi(1)$ is an idempotent of the algebra A^P. For an element $g \in G$, the idempotent ${}^g\alpha \in A$ only depends on the right coset gP. The decomposition $1_A = \sum_{g \in G/P} {}^g\alpha$ of the unity into mutually orthogonal idempotents brings a block decomposition of the matrix algebra A, which is also a decomposition of the $\mathcal{O}(P \times P)$-module A:

$$A \;=\; \bigoplus_{g,h \in G/P} {}^g\alpha \; A \; {}^h\alpha \;=\; \bigoplus_{g,h \in G/P} g(\alpha A \alpha)h^{-1}.$$

By [Pu-1986], the $C_G(Q)$-interior G-algebra $A_Q = \mathrm{Br}_Q(A)$ admits an extension to a G-interior algebra, which makes A_Q a $k(G \times G)$-module. Consider the idempotent $\alpha_Q = \mathrm{br}_Q(\alpha) \in (A_Q)^P$. The G-interior structure on the algebra A_Q induces a P-interior structure on the subalgebra $\alpha_Q A_Q \alpha_Q$. The decomposition $1_{A_Q} = \sum_{g \in G/P} {}^g\alpha_Q$ of the unity into mutually orthogonal idempotents brings a decomposition of the $k(P \times P)$-module A_Q:

$$A_Q \;=\; \bigoplus_{g,h \in G/P} {}^g\alpha_Q \; A_Q \; {}^h\alpha_Q \;=\; \bigoplus_{g,h \in G/P} g(\alpha_Q A_Q \alpha_Q)h^{-1}.$$

On the other hand, we know from [Da-1978, Theorem 4.15] that the P-algebra $\mathrm{Br}_Q(S)$ is a matrix algebra over the field k and extends uniquely to a P-interior algebra. Thus the Brauer quotient $\mathrm{Br}_Q(S)$ can be considered as a $k(P \times P)$-module. We consider the G-interior algebra $B = \mathrm{Ind}_P^G \mathrm{Br}_Q(S)$. As a $k(G \times G)$-module, we have

$$B \;=\; kG \otimes_{kP} \mathrm{Br}_Q(S) \otimes_{kP} kG.$$

The map $\psi : \mathrm{Br}_Q(S) \to B, s \mapsto 1_G \otimes s \otimes 1_G$ is an embedding of algebras. It induces an isomorphism of P-algebras $\mathrm{Br}_Q(S) \simeq \beta B \beta$, where $\beta = \psi(1)$ is an idempotent of the algebra B^P. For an element $g \in G$, the idempotent ${}^g\beta \in B$ only depends on the right coset gP. The decomposition $1_B = \sum_{g \in G/P} {}^g\beta$ of the unity into mutually orthogonal idempotents brings a decomposition of the $k(P \times P)$-module B:

$$B = \bigoplus_{g,h \in G/P} {}^g\beta \; B \; {}^h\beta = \bigoplus_{g,h \in G/P} g(\beta B \beta)h^{-1}.$$

The Brauer functor Br_Q sends the map $\phi : S \to A$ to a morphism of algebras $\phi_Q = \mathrm{Br}_Q(\phi) : \mathrm{Br}_Q(S) \to \mathrm{Br}_Q(A)$. Since the map ϕ induces an isomorphism of P-algebras $S \simeq \alpha A \alpha$, the map ϕ_Q induces an isomorphism of P-algebras $\mathrm{Br}_Q(S) \simeq \alpha_Q A \alpha_Q$. It follows from the uniqueness of the P-interior structure on $\mathrm{Br}_Q(S)$ that this is an isomorphism of P-interior algebra, hence of $k(P \times P)$-module. Thus $\phi_Q : \mathrm{Br}_Q(S) \to A_Q$ is a morphism of $k(P \times P)$-modules. By the universal property of induced modules, there exists a unique morphism of $k(G \times G)$-modules $\Phi : B \to A_Q$ such that $\Phi \circ \psi = \phi_Q$.

By construction, we have $\Phi(\beta) = \alpha_Q$ and Φ induces an isomorphism of P-interior algebras $\beta B \beta \simeq \alpha_Q A_Q \alpha_Q$. Since Φ is a morphism of $k(G \times G)$-modules, this implies that Φ induces an isomorphism of $k(P \times P)$-modules

$g(\beta B\beta)h^{-1} \simeq g(\alpha_Q A_Q \alpha_Q)h^{-1}$ for any two elements g, h in the group G. Therefore, it follows from the above decompositions that $\Phi : B \to A_Q$ is an isomorphism of $k(P\times P)$-modules, hence an isomorphism of $k(G\times G)$-modules. Finally, the definition of the multiplication law in induced interior algebras implies that Φ is an isomorphism of algebras. This completes the proof of the lemma. □

Lemma 3.14. *With the notations and assumptions of Lemma 3.13, let i be an idempotent of the algebra $(\mathcal{O}G)^P$. Write $i_Q = \mathrm{br}_Q(i)$. The isomorphism Φ induces an isomorphism of G-interior algebras*

$$\Phi_i : kGi_Q \otimes_{kP} \mathrm{Br}_Q(\mathrm{End}_k(V)) \otimes_{kP} i_Q kG \;\to\; \mathrm{Br}_Q(\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{End}_k(V) \otimes_{\mathcal{O}P} i\mathcal{O}G).$$

Proof. We keep the notations of the proof of Lemma 3.13. On the one hand, we consider the idempotent $u = \mathrm{Tr}_P^G(i \otimes 1_S \otimes i) = \mathrm{Tr}_P^G(i\alpha i)$ of the algebra A^G, and the idempotent $u_Q = \mathrm{br}_Q(u) = \mathrm{Tr}_P^G(i_Q\alpha_Q i_Q)$ of the algebra $(A_Q)^G$. A direct computation in the induced algebra A yields

$$uAu = \mathcal{O}Gi\otimes_{\mathcal{O}P}S\otimes_{\mathcal{O}P}i\mathcal{O}G, \quad \text{so that} \quad u_QA_Qu_Q \simeq \mathrm{Br}_Q(\mathcal{O}Gi\otimes_{\mathcal{O}P}S\otimes_{\mathcal{O}P}i\mathcal{O}G).$$

On the other hand, we consider the idempotent $v = \mathrm{Tr}_P^G(i_Q \otimes 1_{\mathrm{Br}_Q(S)} \otimes i_Q) = \mathrm{Tr}_P^G(i_Q\beta i_Q)$ of the algebra B^G. The same computation brings

$$vBv = kGi_Q \otimes_{kP} \mathrm{Br}_Q(S) \otimes_{kP} i_QkG.$$

We know that $\Phi(\beta) = \alpha_Q$ and Φ is a morphism of $k(G\times G)$-modules, so we have $\Phi(i_Q\beta i_Q) = i_Q\alpha_Q i_Q$. Moreover, Φ is a morphism of G-algebras, so it commutes with the relative trace map and we obtain $\Phi(v) = u_Q$. Finally, $\Phi : B \to A_Q$ is an isomorphism of G-interior algebras, so it induces an isomorphism of G-interior algebras $\Phi_i : vBv \to u_QA_Qu_Q$. This proves the lemma. □

We are now ready to state the main result of this section. If P is a p-group, Q a subgroup of P and V an endopermutation $\mathcal{O}P$-module, we denote by $V(Q)$ a $kN_P(Q)$-module such that there is an isomorphism of $N_P(Q)$-algebras $\mathrm{End}_k(V(Q)) \simeq \mathrm{Br}_Q(\mathrm{End}_{\mathcal{O}}(V))$, *i.e.*, a Q-slashed module attached to V.

Theorem 3.15. *Let G be a finite group and e be a block of the group algebra $\mathcal{O}G$. Let M be a Brauer-friendly $\mathcal{O}Ge$-module. Let (Q, e_Q) be an e-subpair of the group G, and H be a subgroup of G such that $C_G(Q) \leqslant H \leqslant N_G(Q, e_Q)$.*

(i) There exists a Brauer-friendly $kH\bar{e}_Q$-module $M(Q, e_Q)$ and an isomorphism of $C_G(Q)$-interior H-algebras

$$\mathrm{Br}_{(Q,e_Q)}(\mathrm{End}_{\mathcal{O}}(M)) \ \simeq \ \mathrm{End}_k(M(Q, e_Q)).$$

(ii) The isomorphism class of the $kH\bar{e}_Q$-module $M(Q, e_Q)$ is determined by the choice of an H-interior structure on the algebra $\mathrm{Br}_{(Q,e_Q)}(\mathrm{End}_{\mathcal{O}}(M))$. This choice is unique up to twisting by a linear character $\chi : H/C_G(Q) \to k^\times$.

(iii) If $(R, \bar{e}_R, W)$ is a source triple of an indecomposable direct summand of the $kH\bar{e}_Q$-module $M(Q, e_Q)$, then there exists a source triple (P, e_P, V) of an indecomposable direct summand of the $\mathcal{O}Ge$-module M such that the subpair (P, e_P) contains both (Q, e_Q) and (R, e_R), and that the kR-module W is isomorphic to a direct summand of the restriction $\mathrm{Res}_R^{N_P(Q)} V(Q)$.

Proof. The Brauer quotient $\mathrm{Br}_{(Q,e_Q)}(\mathrm{End}_k(M)) \simeq \mathrm{Br}_Q(\mathrm{End}_k(e_Q M))$ is a $C_G(Q)$-interior $N_G(Q, e_Q)$-algebra, hence a $C_H(Q)$-interior H-algebra. We know from Lemma 3.11 (i) that the $\mathcal{O}H$-module $e_Q M$ admits the decomposition $e_Q M = L \oplus L'$, where L is a Brauer-friendly $\mathcal{O}He_Q$-module and L' is a direct sum of indecomposable $\mathcal{O}H$-modules with vertices that do not contain the normal p-subgroup Q. Thus the embedding of L into $e_Q M$ induces an isomorphism of $C_H(Q)$-interior H-algebras $\mathrm{Br}_Q(\mathrm{End}_{\mathcal{O}}(L)) \simeq \mathrm{Br}_Q(\mathrm{End}_{\mathcal{O}}(e_Q M))$.

We know from Lemma 3.11 (ii) that the restriction $\mathrm{Res}_Q^H L$ is an endopermutation $\mathcal{O}Q$-module. By [Da-1978, Theorem 4.15] and [Pu-1986], the $C_H(Q)$-interior H-algebra $\mathrm{Br}_Q(\mathrm{End}_k(L))$ extends to an H-interior algebra, and there exists a $kH\bar{e}_Q$-module $L(Q)$ such that the H-interior algebra $\mathrm{End}_k(L(Q))$ is isomorphic to $\mathrm{Br}_Q(\mathrm{End}_k(L))$. The H-interior structure on the algebra $\mathrm{Br}_Q(\mathrm{End}_k(L))$ is unique up to twisting by a linear character $\chi : H/C_H(Q) \to k^\times$.

By construction, the p-subgroup Q of $N_G(Q, e_Q)$ acts trivially on the Brauer quotient $\mathrm{Br}_{(Q,e_Q)}(\mathrm{End}_k(M))$. So we may extend $L(Q)$ to a $kQH\bar{e}_Q$-module with a trivial action of Q. This extension is Brauer-friendly if, and only if, the $kH\bar{e}_Q$-module $L(Q)$ is Brauer-friendly. Thus, from now on, we may suppose

$QC_G(Q) \leqslant H \leqslant N_G(Q, e_Q)$.

Let (D, e_D) be a maximal e_Q-subpair of the group H, and i be a source idempotent of the block e_Q with respect to the maximal subpair (D, e_D). Write $i_Q = \mathrm{br}_Q(i)$. Since the $\mathcal{O}He_Q$-module L is Brauer-friendly, there exists an endopermutation $\mathcal{O}D$-module V that is fusion-stable in H with respect to the subpair (D, e_D), and such that L is isomorphic to a direct summand of the $\mathcal{O}Ge$-module $X = \mathcal{O}Gi \otimes_{\mathcal{O}D} V$. Let $V(Q)$ be a kD-module such that there is an isomorphism of D-algebras $\mathrm{Br}_Q(\mathrm{End}_{\mathcal{O}}(V)) \simeq \mathrm{End}_k(V(Q))$. The slashed module $V(Q)$ is an endopermutation kD-module that is fusion stable in the group H with respect to the subpair $(D, \bar{e}_D)$.

Since the e_Q-subpair (Q, e_Q) is normal in the group H, it appears from the proof of Lemma 3.11 (ii) that the restriction $\mathrm{Res}^H_Q \mathrm{Ind}^H_D V$ is an endopermutation $\mathcal{O}Q$-module. Thus we may apply Lemma 3.13 and Lemma 3.14, which essentially state that the slashed module $X(Q)$ is isomorphic to the $kH\bar{e}_Q$-module $kHi_Q \otimes_{kD} V(Q)$. We have $\bar{e}_D\,\mathrm{br}_D(i_Q) \neq 0$, and the idempotent $\mathrm{br}_R(i_Q)$ lies in a single block of the algebra $kC_H(R)$ for any subgroup R of D. So, by Lemma 3.10, the $kH\bar{e}_Q$-module $X(Q)$ is Brauer-friendly. *A fortiori*, the direct summand $L(Q)$ is Brauer-friendly. A more precise study of the proof of Lemma 3.10 brings the statement in (iii). □

We say that the $kH\bar{e}_Q$-module $M(Q, e_Q)$ of Theorem 3.15 is a (Q, e_Q)-slashed module attached to the $\mathcal{O}Ge$-module M (with respect to the local subgroup H). The notation $M(Q, e_Q)$ will always stand for a (Q, e_Q)-slashed module attached to M. If $\chi : H/C_G(Q) \to k^\times$ is a linear character, then the twisted module $\chi_* M(Q, e_Q)$ is another (Q, e_Q)-slashed module attached to M. Notice that this twisted module might be isomorphic to the $kH\bar{e}_Q$-module $M(Q, e_Q)$. In general, there is no canonical way of choosing among the various isomorphism classes of (Q, e_Q)-slashed modules attached to the $\mathcal{O}Ge$-module M.

The next lemma deals with the transitivity of the slash construction.

Lemma 3.16. *Let M be a Brauer-friendly $\mathcal{O}Ge$-module, and let (Q, e_Q) be an e-subpair of the group G.*

(i) If (R, e_R) is an e-subpair such that $(Q, e_Q) \triangleleft (R, e_R)$, then there is a natural

isomorphism of $C_G(R)$-interior $N_G(Q,R,e_R)$-algebras

$$\mathrm{Br}_{(R,e_R)}(\mathrm{End}_{\mathcal{O}}(M)) \;\to\; \mathrm{Br}_{(R,e_R)}\circ\mathrm{Br}_{(Q,e_Q)}(\mathrm{End}_{\mathcal{O}}(M)).$$

Up to replacing one of the slashed modules $M(R,e_R)$ and $M(Q,e_Q)(R,e_R)$ by a twisted form, there is an isomorphism of $kN_G(Q,R,e_R)\bar{e}_R$-modules

$$M(R,e_R) \;\simeq\; M(Q,e_Q)(R,e_R).$$

(ii) *If g is an element of the group G, then there is a natural isomorphism of $C_G(Q)$-interior $N_G(Q,e_Q)$-algebras*

$$\mathrm{Br}_{(Q,e_Q)}(\mathrm{End}_{\mathcal{O}}(M)) \;\to\; {}^{g^{-1}}(\mathrm{Br}_{{}^g(Q,e_Q)}(\mathrm{End}_{\mathcal{O}}(M)))$$

Up to replacing one of the slashed modules $M(Q,e_Q)$ and $M({}^g(Q,e_Q))$ by a twisted form, there is an isomorphism of $kN_G(Q,e_Q)\bar{e}_Q$-modules

$$M(Q,e_Q) \;\simeq\; g^{-1}\,M({}^g(Q,e_Q)).$$

If ${}^g(Q,e_Q) = (Q,e_Q)$, then no twisting is needed to obtain this isomorphism.

Proof. We need to compare the algebras $\mathrm{Br}_R(\mathrm{End}_k(e_RM))$ and $\mathrm{Br}_R\circ\mathrm{Br}_Q(\mathrm{End}_k(e_RM))$. Up to replacing G by $N_G(Q,R,e_R)$ and M by e_RM, we may suppose that the p-subgroups $Q\leqslant R$ are normal in the group G, and that $e=e_Q=e_R$.

Let $M=M_1\oplus\cdots\oplus M_n$ be a Krull-Schmidt decomposition of the kGe-module M. For each integer $l\in\{1,\ldots,n\}$, let (P_l,e_l,V_l) be a source triple of the indecomposable direct summand M_l. If the vertex P_l does not contain the normal p-subgroup R, then we have $\mathrm{Br}_R(\mathrm{End}_k(M_l))=0$. Moreover, we know from the proof of Theorem 3.15 that any direct summand of a Q-slashed module $M_l(Q)$ admits a vertex that is contained in P_l. So this vertex does not contain R, and we obtain $\mathrm{Br}_R\circ\mathrm{Br}_Q(\mathrm{End}_k(M_l))=0$. Let M' be the direct sum of the direct summands M_l such that $R\leqslant P_l$. Then the natural embeddings $\mathrm{Br}_R(\mathrm{End}_k(M'))\to\mathrm{Br}_R(\mathrm{End}_k(M))$ and $\mathrm{Br}_R\circ\mathrm{Br}_Q(\mathrm{End}_k(M'))\to\mathrm{Br}_R\circ\mathrm{Br}_Q(\mathrm{End}_k(M))$ are isomorphism. Moreover, by Lemma 3.11 (ii), the restriction $\mathrm{Res}^G_R M'$ is an endopermutation kR-module, so the natural map $\mathrm{Br}_R(\mathrm{End}_k(M'))\to\mathrm{Br}_R\circ\mathrm{Br}_Q(\mathrm{End}_k(M'))$ is an isomorphism. This proves (i).

The proof of (ii) follows from the unicity statement in Theorem 3.15. If ${}^g(Q, e_Q) = (Q, e_Q)$, then the element g lies in the normaliser $N_G(Q, e_Q)$. Since the slashed module $M(Q, e_Q)$ is a $kN_G(Q, e_Q)$-module, the multiplication by g^{-1} is an isomorphism $M(Q, e_Q) \simeq g^{-1}M(Q, e_Q)$. □

The transitivity of the slash construction brings the following straightforward consequence.

Corollary 3.17. *Let M be an indecomposable Brauer-friendly $\mathcal{O}Ge$-module, and (Q, e_Q) be an e-subpair of the group G. A slashed module $M(Q, e_Q)$ is non-zero if, and only if, the e-subpair (Q, e_Q) is contained in a vertex subpair of the indecomposable module M.*

Once again, it is interesting to compare the results of this section with those of [Li-2013, Section 3]. Let M be a Brauer-friendly $\mathcal{O}Ge$-module. Let D be a defect group of the block e and $i \in (\mathcal{O}Ge)^D$ be a source idempotent of the block e. Let $A = i\mathcal{O}Gi$ be the corresponding source algebra of the block e. Then the A-module iM in an endopermutation $\mathcal{O}D$-module. Let Q be a p-subgroup of D, and let e_Q be the unique block of the algebra $\mathcal{O}C_G(Q)$ such that $\bar{e}_Q i_Q \neq 0$, where $i_Q = \mathrm{br}_Q(i)$. The Brauer quotient $\mathrm{Br}_Q(A)$ is naturally isomorphic to the $C_D(Q)$-interior algebra $A_Q = i_Q kC_G(Q) i_Q$.

Consider a slashed module $M(Q, e_Q)$ as a $kC_G(Q)\bar{e}_Q$-module. The slash construction is additive, so the direct summand $i_Q M(Q, e_Q)$ is isomorphic to the A_Q-module that Linckelmann would denote by $\mathrm{Defres}^D_{QC_D(Q)/Q}\, iM$. Suppose moreover that the subpair (Q, e_Q) is fully centralised in D, *i.e.*, that the centraliser $C_D(Q)$ is a defect group of the block e_Q in the group $C_G(Q)$. Then the algebra A_Q is a source algebra of the block e_Q, so the slashed module $M(Q, e_Q)$ can be retrieved from Linckelmann's construction: we have $M(Q, e_Q) \simeq kC_G(Q)i_Q \otimes_{A_Q} \mathrm{Defres}^D_{QC_D(Q)/Q}\, iM$. The fact that $M(Q, e_Q)$ is a Brauer-friendly $kC_G(Q)\bar{e}_Q$-module is then a consequence of Lemma 3.12.

3.4 Localisation functors

A priori, the slash construction is additive but not functorial. In this section, we show how this flaw can be partially mended. We define two types of "slash functors". The first one deals with p-groups and endopermutation modules. The second one deals with general finite groups and Brauer-friendly modules.

Definition 3.18. Let Q be a finite p-group and ${}_{\mathcal{O}Q}\mathbf{M}$ be a full subcategory of the category ${}_{\mathcal{O}Q}\mathbf{Mod}$. Let $Sl : {}_{\mathcal{O}Q}\mathbf{M} \to {}_{k}\mathbf{Mod}$ be a functor. We say that Sl is a Q-slash functor if the map $\mathrm{Hom}_{\mathcal{O}Q}(L, M) \to \mathrm{Hom}_k(Sl(L), Sl(M)), u \mapsto Sl(u)$, factors through an isomorphism

$$\mathrm{Br}_{\Delta Q}(\mathrm{Hom}_{\mathcal{O}}(L, M)) \;\simeq\; \mathrm{Hom}_k(Sl(L), Sl(M)).$$

for any two objects L, M in the category ${}_{\mathcal{O}Q}\mathbf{M}$.

Theorem 3.19. *Let Q be a finite p-group and ${}_{\mathcal{O}Q}\mathbf{M}$ be a full subcategory of the category ${}_{\mathcal{O}Q}\mathbf{Mod}$. Suppose that the subcategory ${}_{\mathcal{O}Q}\mathbf{M}$ is closed under finite direct sums and that any object M of the category ${}_{\mathcal{O}Q}\mathbf{M}$ decomposes as $M = M' \oplus M''$, where M' is an endopermutation $\mathcal{O}Q$-module and M'' is a non-capped $\mathcal{O}Q$-module.*

(i) There exists a Q-slash functor $\mathrm{Sl}_Q : {}_{\mathcal{O}Q}\mathbf{M} \to {}_{k}\mathbf{Mod}$.

(ii) If $Sl : {}_{\mathcal{O}Q}\mathbf{M} \to {}_{k}\mathbf{Mod}$ is another Q-slash functor, then there exists an isomorphism of functors $\mathrm{Sl}_Q \simeq Sl$, and this isomorphism is unique up to scalar multiplication.

Proof. It may happen that no indecomposable direct summand of an object of ${}_{\mathcal{O}Q}\mathbf{M}$ has vertex Q. Then $\mathrm{Br}_{\Delta Q}(\mathrm{Hom}_{\mathcal{O}}(L, M)) = 0$ for any two objects L, M of ${}_{\mathcal{O}Q}\mathbf{M}$, so we can take Sl_Q to be a zero functor. In that case, the unicity statement is obvious.

We now suppose that there exists a capped indecomposable $\mathcal{O}Q$-module V such that V is a direct summand of an object of the category ${}_{\mathcal{O}Q}\mathbf{M}$. We fix the $\mathcal{O}Q$-module V until the end of the proof. Let M be an object of ${}_{\mathcal{O}Q}\mathbf{M}$. We consider $\mathrm{Hom}_{\mathcal{O}}(V, M)$ as an $\mathcal{O}\Delta Q$-module and we set $\mathrm{Sl}_Q(M) = \mathrm{Br}_{\Delta Q}(\mathrm{Hom}_{\mathcal{O}}(V, M))$. Let L, M be two object in ${}_{\mathcal{O}Q}\mathbf{M}$ and $u : L \to M$ be a morphism of $\mathcal{O}Q$-modules. Then the map $\mathrm{Hom}_{\mathcal{O}}(V, u) : \mathrm{Hom}_{\mathcal{O}}(V, L) \to$

$\mathrm{Hom}_{\mathcal{O}}(V,M)$ is a morphism of $\mathcal{O}\Delta Q$-modules, and we set $\mathrm{Sl}_Q(u) = \mathrm{Br}_{\Delta Q}(\mathrm{Hom}_{\mathcal{O}}(V,u))$: $\mathrm{Sl}_Q(L) \to \mathrm{Sl}_Q(M)$. This clearly defines a functor $\mathrm{Sl}_Q : {}_{\mathcal{O}Q}\mathbf{M} \to {}_k\mathbf{Mod}$.

To prove that Sl_Q is a Q-slash functor, we consider two objects L, M in ${}_{\mathcal{O}Q}\mathbf{M}$. The functor Sl_Q defines a $\mathcal{O}$-linear map

$$\mathrm{Sl}_Q^{L,M} : (\mathrm{Hom}_{\mathcal{O}}(L,M))^{\Delta Q} \to \mathrm{Hom}_k(\mathrm{Br}_{\Delta Q}(\mathrm{Hom}_{\mathcal{O}}(V,L)), \mathrm{Br}_{\Delta Q}(\mathrm{Hom}_{\mathcal{O}}(V,M))).$$

which factors through a k-linear map

$$\Phi : \mathrm{Br}_{\Delta Q}(\mathrm{Hom}_{\mathcal{O}}(L,M)) \to \mathrm{Hom}_k(\mathrm{Br}_{\Delta Q}(\mathrm{Hom}_{\mathcal{O}}(V,L)), \mathrm{Br}_{\Delta Q}(\mathrm{Hom}_{\mathcal{O}}(V,M))).$$

We need to prove that Φ is an isomorphism. The category ${}_{\mathcal{O}Q}\mathbf{M}$ is closed under direct sums, and the capped indecomposable direct summands of any object in ${}_{\mathcal{O}Q}\mathbf{M}$ must be compatible endopermutation $\mathcal{O}Q$-modules. Thus V is a capped indecomposable endopermutation $\mathcal{O}Q$-module. The $\mathcal{O}Q$-modules L and M decompose as $L = L' \oplus L''$ and $M = M' \oplus M''$, where L', M' are direct sums of copies of V and L'', M'' are direct sums of non-capped indecomposable $\mathcal{O}Q$-modules. This implies $\mathrm{Br}_{\Delta Q}(\mathrm{Hom}_{\mathcal{O}}(V,L'')) = \mathrm{Br}_{\Delta Q}(\mathrm{Hom}_{\mathcal{O}}(V,M'')) = 0$. Hence the above decompositions of L and M induce isomorphisms

$$\mathrm{Br}_{\Delta Q}(\mathrm{Hom}_{\mathcal{O}}(L',M')) \simeq \mathrm{Br}_{\Delta Q}(\mathrm{Hom}_{\mathcal{O}}(L,M)) \quad \textit{etc.}$$

As a consequence, we may now suppose $L = L'$ and $M = M'$. Then L and M are finite direct sums of copies of V, so it is enough to prove that Φ is an isomorphism in the case $L = M = V$, which is trivial since $\mathrm{Br}_{\Delta Q}(\mathrm{Hom}_{\mathcal{O}}(V,V)) = \mathrm{Br}_Q(\mathrm{End}_{\mathcal{O}}(V)) \simeq k$. It follows that Sl_Q is a Q-slash functor.

We now consider another Q-slash functor $Sl : {}_{\mathcal{O}Q}\mathbf{M} \to {}_k\mathbf{Mod}$. It may happen that the $\mathcal{O}Q$-module V itself is not an object of the category ${}_{\mathcal{O}Q}\mathbf{M}$. So we choose an object X of ${}_{\mathcal{O}Q}\mathbf{M}$ that admits V as a direct summand, and an idempotent $i \in \mathrm{End}_{\mathcal{O}Q}(X)$ such that $V = iX$. The functor Sl defines a map

$$Sl^X : \mathrm{End}_{\mathcal{O}}(X) \ \to \ \mathrm{End}_k(Sl(X)).$$

We set $j = Sl^X(i) \in \mathrm{End}_k(Sl(X))$ and $W = j\,Sl(X)$. The map Sl^X induces an isomorphism $\mathrm{Br}_Q(\mathrm{End}_{\mathcal{O}}(X)) \simeq \mathrm{End}_k(Sl(X))$, which sends the idempotent

$\mathrm{br}_Q(i)$ to j. So the map Sl^X induces an isomorphism

$$\mathrm{Br}_Q(\mathrm{End}_{\mathcal{O}}(V)) \;\simeq\; \mathrm{br}_Q(i)\ \mathrm{Br}_Q(\mathrm{End}_{\mathcal{O}}(X))\ \mathrm{br}_Q(i) \;\simeq\; \mathrm{End}_k(W),$$

which implies that W is a 1-dimensional vector space over the field k.

Let M be an object of the category ${}_{\mathcal{O}Q}\mathbf{M}$. The map $Sl^{X,M} : \mathrm{Hom}_{\mathcal{O}}(X, M) \to \mathrm{Hom}_k(Sl(X), Sl(M))$ induces an isomorphism

$$\phi_M : \mathrm{Br}_{\Delta Q}(\mathrm{Hom}_{\mathcal{O}}(V, M)) \;\to\; \mathrm{Hom}_k(W, Sl(M)),$$

which is natural in M. We may now choose a non-zero element $w \in W$. This brings a natural isomorphism

$$\zeta_M : \mathrm{Hom}_k(W, Sl(M)) \;\to\; Sl(M),\ u \mapsto u(w).$$

By definition, we have $\mathrm{Sl}_Q(M) = \mathrm{Br}_{\Delta Q}(\mathrm{Hom}_{\mathcal{O}}(V, M))$. So we obtain a natural isomorphism $\zeta_M \circ \phi_M : \mathrm{Sl}_Q(M) \to Sl(M)$. We set $i_Q = \mathrm{Sl}_Q^X(i) \in \mathrm{End}_k(\mathrm{Sl}_Q(X))$ and $V_Q = i_Q\,\mathrm{Sl}_Q(X)$, a 1-dimensional k-vector space. If $\xi : \mathrm{Sl}_Q \to Sl$ is another isomorphism of functors, then the map $\xi_X : \mathrm{Sl}_Q(X) \to Sl(X)$ induces an isomorphism $f : V_Q \to W$, which is unique up to scalar multiplication. The correspondence $\xi \mapsto f$ is one-to-one, so the natural isomorphism ξ is unique up to scalar multiplication. □

We now consider Q as a p-subgroup of a larger finite group, and we take subpairs into account.

Definition 3.20. Let G be a finite group, e be a block of the group G, and ${}_{\mathcal{O}Ge}\mathbf{M}$ be a subcategory of the category ${}_{\mathcal{O}Ge}\mathbf{Mod}$. Let (Q, e_Q) be an e-subpair of the group G, and H be a subgroup of G such that $C_G(Q) \leqslant H \leqslant N_G(Q, e_Q)$. A (Q, e_Q)-slash functor $Sl : {}_{\mathcal{O}Ge}\mathbf{M} \to {}_{kH\bar{e}_Q}\mathbf{Mod}$ is defined by the following data:

- for each object M of the category ${}_{\mathcal{O}Ge}\mathbf{M}$, a $kH\bar{e}_Q$-module $Sl(M)$;
- for each pair L, M of objects of the category ${}_{\mathcal{O}Ge}\mathbf{M}$, a map

$$Sl^{L,M} : \mathrm{Hom}_{\mathcal{O}Q}(L, M) \to \mathrm{Hom}_k(Sl(L), Sl(M));$$

such that

- for any two objects L, M of the category ${}_{\mathcal{O}Ge}\mathbf{M}$, the map $Sl^{L,M}$ factors through an isomorphism of $k(C_G(Q) \times C_G(Q))\Delta H$-modules

$$\mathrm{Br}_{(\Delta Q, e_Q \otimes e_Q^{\circ})}(\mathrm{Hom}_{\mathcal{O}}(L, M)) \to \mathrm{Hom}_k(Sl(L), Sl(M));$$

- $Sl^{M,M}(1_{\mathrm{End}_{\mathcal{O}}(M)}) = 1_{\mathrm{End}_k(Sl(M))}$ for any object M of the category ${}_{\mathcal{O}Ge}\mathbf{M}$;

- $Sl^{L,N}(v \circ u) = Sl^{M,N}(v) \circ Sl^{L,M}(u)$ for any three objects L, M, N of the category ${}_{\mathcal{O}Ge}\mathbf{M}$ and any two morphisms of $\mathcal{O}Q$-modules $u : L \to M$, $v : M \to N$.

Notice that the (Q, e_Q)-slash functor Sl is not really a functor from the category ${}_{\mathcal{O}Ge}\mathbf{M}$ to the category ${}_{kH\bar{e}_Q}\mathbf{Mod}$. Actually, it is more than that. For any two objects L, M in ${}_{\mathcal{O}Ge}\mathbf{M}$, the morphism of $\mathcal{O}(C_G(Q) \times C_G(Q))\Delta H$-modules $Sl^{L,M} : \mathrm{Hom}_{\mathcal{O}Q}(L, M) \to \mathrm{Hom}_k(Sl(L), Sl(M))$ induces a map $\tilde{Sl}^{L,M}$: $\mathrm{Hom}_{\mathcal{O}G}(L, M) \to \mathrm{Hom}_{kH}(Sl(L), Sl(M))$. Hence the (Q, e_Q)-slash functor Sl induces an actual functor

$$\tilde{Sl} : {}_{\mathcal{O}Ge}\mathbf{M} \to {}_{kH\bar{e}_Q}\mathbf{Mod}.$$

Remember from Theorem 3.15 that a (Q, e_Q)-slashed module is well-defined up to twisting by a linear character of the quotient group $H/C_G(Q)$. The same will be true for slash functors, so we need to define the twisted forms of a (Q, e_Q)-slash functor. Let $Sl : {}_{\mathcal{O}Ge}\mathbf{M} \to {}_{kH\bar{e}_Q}\mathbf{Mod}$ be a (Q, e_Q)-slash functor, and let $\chi : H/C_G(Q) \to k^{\times}$ be a linear character. If M is an object of the category ${}_{\mathcal{O}Ge}\mathbf{M}$, then we set $\chi_* Sl(M) = Sl(M)$ as a k-vector space, and we endow $\chi_* Sl(M)$ with the action $\cdot_{\chi}$ of the group H defined by $h \cdot_{\chi} m = h \cdot \chi(h) m$ for any $h \in H$ and $m \in Sl(M)$, where the single dot stands for the preexisting action of the group H on the module $Sl(M)$. If L, M are two objects of ${}_{\mathcal{O}Ge}\mathbf{M}$ and $u : L \to M$ is a morphism of $\mathcal{O}Q$-modules, then we set $\chi_* Sl^{L,M}(u) = Sl^{L,M}(u)$, considered as a k-linear map $\chi_* Sl(L) \to \chi_* Sl(M)$. This defines another (Q, e_Q)-slash functor $\chi_* Sl : {}_{\mathcal{O}Ge}\mathbf{M} \to {}_{kH\bar{e}_Q}\mathbf{Mod}$. Notice that the functors Sl and $\chi_* Sl$ might be isomorphic. In general, there is no canonical choice among the various twisted forms of a (Q, e_Q)-slash functor.

Theorem 3.21. *Let G be a finite group, e be a block of the group G, and ${}_{\mathcal{O}Ge}\mathbf{M}$ be a Brauer-friendly subcategory of the category ${}_{\mathcal{O}Ge}\mathbf{Mod}$. Let (Q, e_Q) be an e-subpair of the group G, and H be a subgroup of G such that $C_G(Q) \leqslant H \leqslant N_G(Q, e_Q)$.*

(i) There exists a (Q, e_Q)*-slash functor* $\mathrm{Sl}_{(Q,e_Q)} : {}_{\mathcal{O}Ge}\mathbf{M} \to {}_{kH\bar{e}_Q}\mathbf{Mod}$.

(ii) If $Sl : {}_{\mathcal{O}Ge}\mathbf{M} \to {}_{kH\bar{e}_Q}\mathbf{Mod}$ *is another* (Q, e_Q)*-slash functor, then there exists a linear character* $\chi : H/C_G(Q) \to k^\times$ *and a natural isomorphism* $Sl \simeq \chi_* \mathrm{Sl}_{(Q,e_Q)}$. *Moreover this natural isomorphism is unique up to scalar multiplication.*

(iii) There exists a Brauer-friendly subcategory ${}_{kH\bar{e}_Q}\mathbf{M}$ *of the category* ${}_{kH\bar{e}_Q}\mathbf{Mod}$ *that contains the image of every* (Q, e_Q)*-slash functor* $Sl : {}_{\mathcal{O}Ge}\mathbf{M} \to {}_{kH\bar{e}_Q}\mathbf{Mod}$.

Proof. If no indecomposable direct summand of an object of the category ${}_{\mathcal{O}Ge}\mathbf{M}$ admits a vertex subpair that contains the subpair (Q, e_Q), then we may take $\mathrm{Sl}_{(Q,e_Q)}$ to be a zero functor. The theorem follows immediately. We now suppose that there exists an object X of ${}_{\mathcal{O}Ge}\mathbf{M}$ such that $\mathrm{Br}_{(Q,e_Q)}(\mathrm{End}_{\mathcal{O}}(X)) \neq 0$.

Let ${}_{\mathcal{O}Q}\mathbf{M}$ be the full subcategory of ${}_{\mathcal{O}Q}\mathbf{Mod}$ over the essential image of the functor $e_Q \mathrm{Res}^G_Q : {}_{\mathcal{O}Ge}\mathbf{M} \to {}_{\mathcal{O}Q}\mathbf{Mod}$. By Lemma 3.11, the category ${}_{\mathcal{O}Q}\mathbf{M}$ satisfies the assumptions of Theorem 3.19. Thus there exists a Q-slash functor $\mathrm{Sl}_Q : {}_{\mathcal{O}Q}\mathbf{M} \to {}_k\mathbf{Mod}$. For any two objects L, M in ${}_{\mathcal{O}Q}\mathbf{M}$, the functor Sl_Q induces a map $\mathrm{Sl}_Q^{L,M} : \mathrm{Hom}_{\mathcal{O}Q}(L, M) \to \mathrm{Hom}_k(\mathrm{Sl}_Q(L), \mathrm{Sl}_Q(M))$. For any $\mathcal{O}Ge$-module M, we denote by M' the $\mathcal{O}He_Q$-module $e_Q M$ (and accordingly for L, X *etc.*).

By Theorem 3.15, the k-vector space $\mathrm{Sl}_Q(X)$ admits a structure of $kH\bar{e}_Q$-module such that the map $\mathrm{Sl}_Q^X : \mathrm{End}_{\mathcal{O}Q}(X) \to \mathrm{End}_k(\mathrm{Sl}_Q(X))$ is a morphism of $C_G(Q)$-interior H-algebras. Let M be another object of the category ${}_{\mathcal{O}Ge}\mathbf{M}$. Let $u : X' \to X' \oplus M'$ and $v : M' \to X' \oplus M'$ be the inclusion maps. Once again, the k-vector space $\mathrm{Sl}_Q(X' \oplus M')$ admits a structure of $kH\bar{e}_Q$-module such that the map

$$\mathrm{Sl}_Q^{X' \oplus M'} : \mathrm{End}_{\mathcal{O}Q}(X' \oplus M') \ \to \ \mathrm{End}_k(\mathrm{Sl}_Q(X' \oplus M'))$$

is a morphism of $C_G(Q)$-interior H-algebras. There may be more than one such structure, but only one of them makes the embedding $\mathrm{Sl}_Q(u) : \mathrm{Sl}_Q(X') \to \mathrm{Sl}_Q(X' \oplus M')$ a morphism of kH-modules. Then pulling back via the embedding $\mathrm{Sl}_Q(v) : \mathrm{Sl}_Q(M') \to \mathrm{Sl}_Q(X' \oplus M')$ makes $\mathrm{Sl}_Q(M')$ a kH-module. This defines a structure of kH-module on $\mathrm{Sl}_Q(M')$ for any object M of the category ${}_{\mathcal{O}Ge}\mathbf{M}$. Let L, M be two objects of ${}_{\mathcal{O}Ge}\mathbf{M}$. By a similar argument on the endomorphism ring of the direct sum $X' \oplus L' \oplus M'$, we check that the map

$$\mathrm{Sl}_Q^{L',M'} : \mathrm{Hom}_{\mathcal{O}Q}(L', M') \to \mathrm{Hom}_k(\mathrm{Sl}_Q(L'), \mathrm{Sl}_Q(M'))$$

is a morphism of $\mathcal{O}(C_G(Q) \times C_G(Q))\Delta H$-modules. For any object M of the category ${}_{\mathcal{O}Ge}\mathbf{M}$, we set $\mathrm{Sl}_{(Q,e_Q)}(M) = \mathrm{Sl}_Q(M')$, endowed with the structure of $kH\bar{e}_Q$-module that has been chosen. For any two objects L, M of ${}_{\mathcal{O}Ge}\mathbf{M}$ and any morphism of $\mathcal{O}Q$-modules $u : L \to M$, we define $e_Q u e_Q : L' \to M'$ as in Section 1.5, and we set

$$\mathrm{Sl}^{L,M}_{(Q,e_Q)}(u) \;=\; \mathrm{Sl}^{L',M'}_Q(e_Q u e_Q) \;:\; \mathrm{Sl}_{(Q,e_Q)}(L) \;\to\; \mathrm{Sl}_{(Q,e_Q)}(M).$$

This defines a (Q, e_Q)-slash functor $\mathrm{Sl}_{(Q,e_Q)} : {}_{\mathcal{O}Ge}\mathbf{M} \to {}_{kH\bar{e}_Q}\mathbf{Mod}$, and proves (i).

We now consider another (Q, e_Q)-slash functor $Sl : {}_{\mathcal{O}Ge}\mathbf{M} \to {}_{kH\bar{e}_Q}\mathbf{Mod}$. For any two objects L, M of the category ${}_{\mathcal{O}Ge}\mathbf{M}$, there is a unique isomorphism $\Phi_{L,M}$ that makes the following diagram commutative:

$$\begin{array}{ccc} & \mathrm{Hom}_{\mathcal{O}Q}(L, M) & \\ {}^{\mathrm{Sl}^{L,M}_{(Q,e_Q)}}\swarrow & & \searrow^{Sl^{L,M}} \\ \mathrm{Hom}_k(\mathrm{Sl}_{(Q,e_Q)}(L), \mathrm{Sl}_{(Q,e_Q)}(M)) & \xrightarrow{\quad\Phi_{L,M}\quad} & \mathrm{Hom}_k(Sl(L), Sl(M)). \end{array}$$

In particular, $\Phi_{L,M}$ is an isomorphism of $k(C_G(Q) \times C_G(Q))\Delta H$-modules. If $L = M$, then we write Φ_M for $\Phi_{M,M}$, and this Φ_M is an isomorphism of $C_G(Q)$-interior H-algebras. It may happen that Φ_X is not an isomorphism of H-interior algebras. In such a case, we remember that the H-interior structure on the algebra $\mathrm{Br}_{(Q,e_Q)}(\mathrm{End}_{\mathcal{O}}(X))$ is unique up to twisting by a linear character χ of the group $H/C_G(Q)$. Up to replacing the functor $\mathrm{Sl}_{(Q,e_Q)}$ by the twisted form $\chi_* \mathrm{Sl}_{(Q,e_Q)}$, we may therefore suppose that Φ_X is an isomorphism of H-interior algebras. Since we have $\mathrm{Sl}_{(Q,e_Q)}(X) \neq 0$, this is enough to ensure that $\Phi_{L,M}$ is an isomorphism of $k(H \times H)$-modules for any two objects L, M of ${}_{\mathcal{O}Ge}\mathbf{M}$. In particular, Φ_M is an isomorphism of H-interior algebras if $L = M$.

Let $\phi_X : \mathrm{Sl}_{(Q,e_Q)}(X) \to Sl(X)$ be a k-linear isomorphism such that $\Phi_X(u) = \phi_X \circ u \circ \phi_X^{-1}$ for any $u \in \mathrm{End}_k(\mathrm{Sl}_{(Q,e_Q)}(X))$. By the Skolem-Noether theorem, such a ϕ_X exists, and it is unique up to multiplication by a scalar. Let M be another object of the category ${}_{\mathcal{O}Ge}\mathbf{M}$. There is an isomorphism $\psi : \mathrm{Sl}_{(Q,e_Q)}(X \oplus M) \to Sl(X \oplus M)$ such that $\Phi_{X \oplus M}(u) = \psi \circ u \circ \psi^{-1}$ for any $u \in \mathrm{End}_k(\mathrm{Sl}_{(Q,e_Q)}(X \oplus M))$. Once again, this ψ is unique up to scalar multiplication. Via the natural embeddings $\mathrm{Sl}_{(Q,e_Q)}(X) \to \mathrm{Sl}_{(Q,e_Q)}(X \oplus M)$ and $Sl(X) \to Sl(X \oplus M)$, the isomorphism ψ restricts to an isomorphism $\psi_X : \mathrm{Sl}_{(Q,e_Q)}(X) \to Sl(X)$, which is a scalar multiple of ϕ_X. Up to replacing ψ itself by a scalar multiple, we may suppose $\psi_X = \phi_X$. With this assumption, the isomorphism ψ becomes unique. Then we let $\phi_M : \mathrm{Sl}_{(Q,e_Q)}(M) \to Sl(M)$ be the restriction of ψ via the natural embeddings $\mathrm{Sl}_{(Q,e_Q)}(M) \to \mathrm{Sl}_{(Q,e_Q)}(X \oplus M)$ and $Sl(M) \to Sl(X \oplus M)$.

As a consequence, we have $\Phi_{L,M}(u) = \phi_M \circ u \circ \phi_L^{-1}$ for any two objects L, M of the category ${}_{\mathcal{O}Ge}\mathbf{M}$ and any $u \in \mathrm{Hom}_k(\mathrm{Sl}_{(Q,e_Q)}(X), \mathrm{Sl}_{(Q,e_Q)}(M))$. In other words, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Sl}_{(Q,e_Q)}(L) & \xrightarrow{\phi_L} & Sl(L) \\
{\scriptstyle \mathrm{Sl}^{L,M}_{(Q,e_Q)}(v)}\downarrow & & \downarrow{\scriptstyle Sl^{L,M}(v)} \\
\mathrm{Sl}_{(Q,e_Q)}(M) & \xrightarrow{\phi_M} & Sl(M)
\end{array}$$

for any two objects L, M of the category ${}_{\mathcal{O}Ge}\mathbf{M}$ and any morphism of $\mathcal{O}Q$-modules $v : L \to M$. Thus $\phi : \mathrm{Sl}_{(Q,e_Q)} \to Sl$ is exactly what we would like to call an isomorphism of (Q, e_Q)-slash functors, and (ii) is proven.

The statement in (iii) follows from Theorem 3.15 (iii). □

3.5 Lifting direct summands

In this section, we will need the following result, which we take from [BR-0000].

Fact 3.22. *Let $\pi : A \to B$ be a morphism of $\mathcal{O}$-algebras that are finitely generated as $\mathcal{O}$-modules. Let $\mathfrak{a}$ be an ideal of the algebra A. The map π induces a one-to-one correspondence between the conjugacy classes of primitive idempotents of the algebra A that lie in the ideal $\mathfrak{a}$ but not in the kernel* $\ker \pi$ *and the conjugacy classes of primitive idempotents of the algebra $\pi(A)$ that lie in the ideal $\pi(\mathfrak{a})$.*

Lemma 3.23. *Let G be a finite group and e be a block of the group algebra $\mathcal{O}G$. Let ${}_{\mathcal{O}Ge}\mathbf{M}$ be a Brauer-friendly subcategory of the category ${}_{\mathcal{O}Ge}\mathbf{Mod}$, and L be an object of the subcategory ${}_{\mathcal{O}Ge}\mathbf{M}$. Let (Q, e_Q) be an e-subpair of the group G and $\mathrm{Sl}_{(Q,e_Q)} : {}_{kGe}\mathbf{M} \to {}_{kN_G(Q,e_Q)\bar{e}_Q}\mathbf{Mod}$ be a (Q, e_Q)-slash functor. If the $kN_G(Q, e_Q)$-module $M = \mathrm{Sl}_{(Q,e_Q)}(L)$ admits an indecomposable direct summand Y with vertex Q, then the $\mathcal{O}G$-module L admits an indecomposable direct summand X with* *vertex subpair (Q, e_Q) such that the slashed module $\mathrm{Sl}_{(Q,e_Q)}(X)$ is isomorphic to the $kN_G(Q, e_Q)$-module Y.*

Proof. For any element $a \in \mathrm{End}_{\mathcal{O}Q}(L)$, we have

$$\begin{aligned} \mathrm{br}_{(Q,e_Q)} \circ \mathrm{Tr}_Q^G(e_Q\, a\, e_Q) &= \sum_{g \in N_G(Q,e_Q)\backslash G/Q} \mathrm{br}_Q \circ \mathrm{Tr}_{N_{{}^gQ}(Q,e_Q)}^{N_G(Q,e_Q)}(e_Q\, {}^g e_Q\, {}^g a\, {}^g e_Q\, e_Q) \\ &= \mathrm{Tr}_Q^{N_G(Q,e_Q)} \circ \mathrm{br}_{(Q,e_Q)}(a) \end{aligned}$$

This computation proves that the Brauer morphism $\mathrm{br}_{(Q,e_Q)}$ sends the ideal $\mathrm{Tr}_Q^G(\mathrm{End}_{\mathcal{O}Q}(L))$ of the algebra $\mathrm{End}_{\mathcal{O}G}(L)$ onto the ideal $\mathrm{Tr}_Q^{N_G(Q,e_Q)}(\mathrm{End}_k(M))$ of the algebra $\mathrm{End}_{kN_G(Q,e_Q)}(M)$. If Y is a indecomposable direct summand of the $kN_G(Q, e_Q)$-module M, then there is a primitive idempotent v of the algebra $\mathrm{End}_{kN_G(Q,e_Q)}(M)$ such that $Y = vM$. If moreover the direct summand Y has vertex Q, then the idempotent v lies in the ideal $\mathrm{Tr}_Q^{N_G(Q,e_Q)}(\mathrm{End}_k(M))$. By Lemma 3.22, this primitive idempotent can be lifted to a primitive idempotent u of the algebra $\mathrm{End}_{\mathcal{O}G}(L)$ such that u lies in the ideal $\mathrm{Tr}_Q^G(\mathrm{End}_{\mathcal{O}Q}(L))$. Then the submodule $X = uL$ is a relatively Q-projective indecomposable direct summand of the $\mathcal{O}G$-module L. Since $v = \mathrm{br}_{(Q,e_Q)}(u)$, we have $\mathrm{Sl}_{(Q,e_Q)}(X) \simeq v\,\mathrm{Sl}_{(Q,e_Q)}(L) = vM = Y \neq 0$, so (Q, e_Q) is a vertex subpair of X. □

Theorem 3.24. *Let G be a finite group and e be a block of the group algebra $\mathcal{O}G$. Let (P, e_P, V) be a fusion-stable endopermutation source triple of the group G with respect to the block e. Let ${}_{\mathcal{O}Ge}\mathbf{M}$ be a Brauer-friendly subcategory of ${}_{\mathcal{O}Ge}\mathbf{Mod}$ such that every indecomposable $\mathcal{O}Ge$-module with source triple (P, e_P, V) is an object of ${}_{\mathcal{O}Ge}\mathbf{M}$. Let $\mathrm{Sl}_{(P,e_P)} : {}_{kGe}\mathbf{M} \to {}_{kN_G(P,e_P)\bar{e}_P}\mathbf{Mod}$ be a (P, e_P)-slash functor. The slash functor $\mathrm{Sl}_{(P,e_P)}$ induces a one-to-one correspondence between the isomorphism classes of indecomposable $\mathcal{O}Ge$-modules with source triple (P, e_P, V) and the isomorphism classes of indecomposable $kN_G(P, e_P)\bar{e}_P$-modules with vertex P and trivial source,* i.e., *the isomorphism classes of projective indecomposable $k(N_G(P, e_P)/P)\bar{e}_P$-modules.*

Proof. Let X be an indecomposable $\mathcal{O}Ge$-module with source triple (P, e_P, V). By definition of a source triple, the indecomposable $\mathcal{O}Ge$-module X is isomorphic to a direct summand of the $\mathcal{O}Ge$-module $L = e\mathcal{O}Ge_P \otimes_{\mathcal{O}P} V$. Thanks to Lemma 3.2, we may write $L = L_1 \oplus L_2$, where L_1 is a direct sum of indecomposable $\mathcal{O}Ge$-modules with source triple (P, e_P, V), and L_2 is a direct sum of indecomposable modules with vertices strictly contained in the p-group P. By assumption, L_1 is an object of the category ${}_{\mathcal{O}Ge}\mathbf{M}$.

We have $L = e\operatorname{Ind}_{N_G(P,e_P)}^G L'$, where $L' = \mathcal{O}N_G(P,e_P)e_P \otimes_{\mathcal{O}P} V$ is a direct sum of indecomposable $\mathcal{O}N_G(P,e_P)e_P$-modules with source triple (P,e_P,V). It follows from the definition of the Green correspondence that the restriction $\operatorname{Res}_{N_G(P,e_P)}^G L_1$ admits a decomposition

$$\operatorname{Res}_{N_G(P,e_P)}^G L_1 \simeq L' \oplus L'',$$

where L'' is a direct sum of indecomposable $\mathcal{O}N_G(P,e_P)$-modules with vertices that do not contain the p-group P. This decomposition induces an embedding of $N_G(P,e_P)$-interior algebras

$$\phi : \operatorname{End}_{\mathcal{O}}(L') \;\to\; \operatorname{End}_{\mathcal{O}}(L_1),$$

which in turn induces an isomorphism of $C_G(P)$-interior $N_G(P,e_P)$-algebras

$$\phi_P : \operatorname{Br}_P(\operatorname{End}_{\mathcal{O}}(L')) \;\to\; \operatorname{Br}_{(P,e_P)}(\operatorname{End}_{\mathcal{O}}(L_1)) \;\simeq\; \operatorname{End}_k(\operatorname{Sl}_{(P,e_P)}(L_1)).$$

The $C_G(P)$-interior $N_G(P,e_P)$-algebra $\operatorname{Br}_P(\operatorname{End}_{\mathcal{O}}(L'))$ can be extended to an $N_G(P,e_P)$-interior algebra by pulling back the $N_G(P,e_P)$-interior structure of the algebra $\operatorname{End}_k(\operatorname{Sl}_{(P,e_P)}(L_1))$ through the isomorphism ϕ_P. Moreover, by assumption, the endopermutation $\mathcal{O}P$-module V is $N_G(P,e_P)$-stable, and $\operatorname{Br}_P(\operatorname{End}_k(V)) \simeq k$. Thus, by Lemma 3.14, there is an isomorphism of $N_G(P,e_P)$-interior algebras

$$\operatorname{Br}_P(\operatorname{End}_{\mathcal{O}}(L')) \;\simeq\; \operatorname{End}_k(kN_G(P,e_P)\bar{e}_P \otimes_{kP} k),$$

hence, by the Skolem-Noether theorem, an isomorphism of $kN_G(P,e_P)\bar{e}_P$-modules

$$\operatorname{Sl}_{(P,e_P)}(L_1) \;\simeq\; kN_G(P,e_P)\bar{e}_P \otimes_{kP} k.$$

Let Y be an indecomposable $kN_G(P,e_P)\bar{e}_P$-module with vertex P and trivial source. Then Y is isomorphic to an indecomposable direct summand of the $kN_G(P,e_P)\bar{e}_P$-module $kN_G(P,e_P)\bar{e}_P \otimes_{kP} k$, *i.e.*, of $\operatorname{Sl}_{(P,e_P)}(L_1)$.

Let us set $A = \operatorname{End}_{\mathcal{O}}(L_1)$. As in the proof of Lemma 3.23, we know that the Brauer morphism $\operatorname{br}_{(P,e_P)}$ sends the ideal $\operatorname{Tr}_P^G(A^P)$ of the algebra A^G onto the ideal $\operatorname{Tr}_P^{N_G(P,e_P)}(\operatorname{Br}_{(P,e_P)}(A))$ of the algebra $\operatorname{Br}_{(P,e_P)}(A)^{N_G(P,e_P)}$. Moreover, the modules L_1 and Sl are relatively P-projective, so the Brauer morphism $\mathrm{br}_{(P,e_P)}$ induces an epimorphism

$$A^G \;\to\; \mathrm{Br}_{(P,e_P)}(A)^{N_G(P,e_P)}.$$

By Lemma 3.22, we conclude that the Brauer morphism $\mathrm{br}_{(P,e_P)}$ induces a one-to-one correspondence between the conjugacy classes of primitive idempotents of the algebra A that are killed by $\mathrm{br}_{(P,e_P)}$ and the conjugacy classes of primitive idempotents of the algebra $\mathrm{Br}_{(P,e_P)}(A)$. In other words, it induces a one-to-one correspondence between the isomorphism classes of indecomposable direct summands of L_1 and the isomorphism classes of indecomposable direct summands of $\mathrm{Sl}_{(P,e_P)}(L_1)$. This proves the theorem. □

The one-to-one correspondence of Theorem 3.24 may be seen as an instance of the Puig correspondence defined in [Pu-1988a]. With the notation of the lemma, notice that this correspondence depends on the isomorphism class of the slash functor $\mathrm{Sl}_{(P,e_P)}$. This is consistent with what Thévenaz explains in [Th-1995, discussion before Example 26.5].

3.6 An approach with 2-categories

We have proven that the behaviour of the slash functors for Brauer-friendly modules is very similar to that of the Brauer functor for p-permutation modules, at least regarding the properties that we have listed in Fact 3.1. We now go a little deeper into the properties of the Brauer functor, following [Rq-1998]. We sketch a possible generalisation of these to the slash functors, which may motivate future research. To make the notations lighter, we now suppose that $\mathcal{O} = k$. Let G be a finite group and e be a block of the algebra kG. The following 2-category is derived from the category that Rouquier uses in [Rq-1998].

Definition 3.25. Let $\mathbf{T}(G,e)$ be the 2-category defined as follows (the $\mathbf{T}$ stands for *transporter*). An object in $\mathbf{T}(G,e)$ is an e-subpair (P,e_P). A 1-arrow $\phi : (P,e_P) \to (Q,e_Q)$ in $\mathbf{T}(G,e)$ is a triple $((P,e_P),(Q,e_Q),g)$, where $g \in G$ is an element such that ${}^g(P,e_P) \leqslant (Q,e_Q)$. If we have ${}^g(P,e_P) \lhd (Q,e_Q)$, we say that ϕ is a normal 1-arrow. The composition of two 1-arrows $\phi = ((P,e_P),(Q,e_Q),g)$ and $\psi = ((Q,e_Q),(R,e_R),h)$ is the 1-arrow $\psi\phi = ((P,e_P),(R,e_R),hg)$. By definition of the inclusion of e-subpairs, the set of 1-arrows in $\mathbf{T}(G,e)$ is generated by the subset of normal 1-arrows.

If $\phi_1, \phi_2 : (P, e_P) \to (Q, e_Q)$ are the 1-arrows defined respectively by elements $g, h \in G$, then a 2-arrow $u : \phi_1 \to \phi_2$ is an element $c \in C_G(P)$ such that $h = gc$. In other words, there is at most one 2-arrow $\phi_1 \to \phi_2$, and there is one if, and only if, $g^{-1}h \in C_G(P)$. If such a 2-arrow exists, then there exists a 2-arrow $\psi\phi_1 \to \psi\phi_2$ for any 1-arrow ψ such that $\psi\phi_1$ exists, and that there exists a 2-arrow $\phi_1\psi \to \phi_2\psi$ for any 1-arrow ψ such that $\phi_1\psi$ exists. If $\phi_3 : (P, e_P) \to (Q, e_Q)$ is a third 1-arrow, and if there exist 2-arrows $\phi_1 \to \phi_2$ and $\phi_2 \to \phi_3$, then there exists a 2-arrow $\phi_1 \to \phi_3$. Thus the vertical and horizontal compositions of 2-arrows are well-defined. Notice that the 2-arrows are always invertible.

To show the relevance of the 2-category $\mathbf{T}(G,e)$, we need to consider the Brauer category $\mathbf{Br}(G,e)$ as a 2-category, which we denote by $\mathbf{Br}^2(G,e)$ to avoid any confusion. An object in $\mathbf{Br}^2(G,e)$ is an object in $\mathbf{Br}(G,e)$, a 1-arrow in $\mathbf{Br}^2(G,e)$ is an arrow in $\mathbf{Br}(G,e)$, and the only 2-arrows in $\mathbf{Br}^2(G,e)$ are the identity 2-arrows attached to every 1-arrow.

We consider the map $\Pi : \mathbf{T}(G,e) \to \mathbf{Br}^2(G,e)$ defined as follows. The map Π sends an object (P, e_P) in $\mathbf{T}(G,e)$ to the same object in $\mathbf{Br}^2(G,e)$. It sends a 1-arrow $\phi = ((P, e_P), (Q, e_Q), g)$ in $\mathbf{T}(G,e)$ to the 1-arrow $\bar{\phi} : (P, e_P) \to (Q, e_Q)$ in $\mathbf{Br}^2(G,e)$ defined by $\bar{\phi}(x) = {}^g x$ for any $x \in P$. For any two 1-arrows $\phi_1, \phi_2 : (P, e_P) \to (Q, e_Q)$ in $\mathbf{T}(G,e)$, there is a 2-arrow $\phi_1 \to \phi_2$ in $\mathbf{T}(G,e)$ if, and only if, the 1-arrows $\bar{\phi}_1$ and $\bar{\phi}_2$ coincide in $\mathbf{Br}^2(G,e)$. Then the map Π sends the unique 2-arrow $\phi_1 \to \phi_2$ in $\mathbf{T}(G,e)$ to the identity 2-arrow of $\bar{\phi}_1 = \bar{\phi}_2$ in $\mathbf{Br}^2(G,e)$. The following lemma is straightforward.

Lemma 3.26. *The map $\Pi : \mathbf{T}(G,e) \to \mathbf{Br}^2(G,e)$ is an equivalence of 2-categories.*

What we define now may be seen as some kind of representation of the 2-category $\mathbf{T}(G,e)$, hence of the Brauer 2-category $\mathbf{Br}^2(G,e)$. For an e-subpair (P, e_P) of the group G and a subgroup H of G such that $C_G(P) \leqslant H \leqslant N_G(P, e_P)$, we denote by $\bar{H}$ the quotient group PH/P, and by ${}_{k\bar{H}e_P}\mathbf{Perm}$ the category of p-permutation $k\bar{H}e_P$-modules, which we consider as $kPHe_P$-modules on which the normal p-subgroup P acts trivially. Let $\phi : (P, e_P) \to (Q, e_Q)$ be a normal 1-arrow in $\mathbf{T}(G,e)$, defined by an element $g \in G$. We write $\phi(P) = {}^gP$ and $\phi(P,e_P) = {}^g(P,e_P)$. We define a functor

$$\mathrm{Br}_\phi : {}_{k\bar{N}_G(P,e_P)e_P}\mathbf{Perm} \;\to\; {}_{k\bar{N}_G(\phi(P),Q,e_Q)e_Q}\mathbf{Perm},$$

as follows. For any $k\bar{N}_G(P,e_P)e_P$-module M, we set $\mathrm{Br}_\phi(M) = \mathrm{Br}_{(Q,e_Q)}(gM)$, where gM is the $k\bar{N}_G({}^g(P,e_P)){}^ge_P$-module with the same underlying k-vector space as M and the group $N_G({}^g(P,e_P))$ acting via the isomorphism $N_G({}^g(P,e_P)) \to N_G(P,e_P)$, $x \mapsto {}^{g^{-1}}x$. For any two $k\bar{N}_G(P,e_P)e_P$-modules L, M, and any morphism of $k({}^{g^{-1}}Q)$-modules $u : L \to M$, we set $\mathrm{Br}_\phi(u) = \mathrm{Br}_{(Q,e_Q)}(u)$, where $u : gL \to gM$ is considered as a morphism of kQ-modules. The map $u \mapsto \mathrm{Br}_\phi(u)$ induces an isomorphism of $k(C_G(Q) \times C_G(Q))\Delta N_G(\phi(P),Q,e_Q)$-modules

$$\mathrm{Br}_{(\Delta Q, e_Q \otimes e_Q^\circ)}({}^g\operatorname{Hom}_k(L,M)) \;\simeq\; \operatorname{Hom}_k(\mathrm{Br}_\phi(L), \mathrm{Br}_\phi(M)).$$

It is also clearly compatible with the composition of morphisms. Thus Br_ϕ is exactly what we would like to call a ϕ-slash functor. When ϕ is an identity 1-arrow, Br_ϕ is the corresponding identity functor. The following two lemmas are essentially proven in [Rq-1998]; we have only adapted the statements to replace the p-subgroups by e-subpairs.

Lemma 3.27. *Let $\phi_1, \phi_2 : (P,e_P) \to (Q,e_Q)$ be two normal 1-arrows in the 2-category $\mathbf{T}(G,e)$ such that there exists a 2-arrow $\phi_1 \to \phi_2$ in $\mathbf{T}(G,e)$.*

(i) There is a canonical isomorphism of functors

$$\eta_{\phi_2,\phi_1}(M) : \mathrm{Br}_{\phi_1} \;\to\; \mathrm{Br}_{\phi_2}\,.$$

(ii) For any normal 1-arrow $\phi_3 : (P,e_P) \to (Q,e_Q)$ such that there is 2-arrow $\phi_2 \to \phi_3$ in $\mathbf{T}(G,e)$, the following diagram is commutative.

$$\begin{array}{ccccc} & & \mathrm{Sl}_{\phi_2} & & \\ & {\scriptstyle\eta_{\phi_2,\phi_1}}\nearrow & & \searrow{\scriptstyle\eta_{\phi_3,\phi_2}} & \\ \mathrm{Sl}_{\phi_1} & & \xrightarrow[\eta_{\phi_3,\phi_1}]{} & & \mathrm{Sl}_{\phi_3} \end{array}$$

Lemma 3.28. *Let $\phi : (P,e_P) \to (Q,e_Q)$ and $\psi : (Q,e_Q) \to (R,e_R)$ be two normal 1-arrows in the 2-category $\mathbf{T}(G,e)$, such that the composition $\psi\phi : (P,e_P) \to$*

(R, e_R) is again a normal 1-arrow.

(i) For any p-permutation $k\bar{N}_G(P, e_P)e_P$-module M, there exists a canonical isomorphism of $kN(\psi\phi(P), \psi(Q), R, e_R)$-modules

$$\theta_{\psi,\phi}(M) : \mathrm{Br}_{\psi\phi}(M) \;\to\; \mathrm{Br}_\psi \circ \mathrm{Br}_\phi(M),$$

which is natural in M. With a slight abuse, we will write $\theta_{\psi,\phi}$ as a natural isomorphism $\mathrm{Br}_{\psi\phi} \to \mathrm{Br}_\psi \circ \mathrm{Br}_\phi$.

(ii) for any normal 1-arrow $\xi : (R, e_R) \to (S, e_S)$ in $\mathbf{T}(G, e)$ such that the 1-arrows $\xi\psi$ and $\xi\psi\phi$ are also normal, the following diagram is commutative.

$$\begin{array}{ccc}
\mathrm{Br}_{\xi\psi\phi} & \xrightarrow{\theta_{\xi,\psi\phi}} & \mathrm{Br}_\xi \circ \mathrm{Br}_{\psi\phi} \\
{\scriptstyle \theta_{\xi\psi,\phi}}\downarrow & & \downarrow{\scriptstyle \mathbf{1}_{\mathrm{Br}_\xi} \circ \theta_{\psi,\phi}} \\
\mathrm{Br}_{\xi\psi} \circ \mathrm{Br}_\phi & \xrightarrow[\theta_{\xi,\psi} \circ \mathbf{1}_{\mathrm{Br}_\phi}]{} & \mathrm{Br}_\xi \circ \mathrm{Br}_\psi \circ \mathrm{Br}_\phi
\end{array}$$

(iii) For any 1-arrow $\phi' : (P, e_P) \to (Q, e_Q)$ in $\mathbf{T}(G, e)$ such that there exists a 2-arrow $\phi \to \phi'$ (hence the composition $\psi\phi'$ is also normal and there is a 2-arrow $\psi\phi \to \psi\phi'$), the following diagram is commutative.

$$\begin{array}{ccc}
\mathrm{Br}_{\psi\phi} & \xrightarrow{\eta_{\psi\phi',\psi\phi}} & \mathrm{Br}_{\psi\phi'} \\
{\scriptstyle \theta_{\psi,\phi}}\downarrow & & \downarrow{\scriptstyle \theta_{\psi,\phi'}} \\
\mathrm{Br}_\psi \circ \mathrm{Br}_\phi & \xrightarrow[\mathbf{1}_{\mathrm{Br}_\psi} \circ \eta_{\phi',\phi}]{} & \mathrm{Br}_\psi \circ \mathrm{Br}_{\phi'}
\end{array}$$

The similar statement is true for composition on the other side.

Notice that this is not really a representation of the 2-category $\mathbf{T}(G, e)$. Indeed, for a 1-arrow $\phi : (P, e_P) \to (Q, e_Q)$ in $\mathbf{T}(G, e)$, the codomain of the functor Br_ϕ is the category ${}_{k\bar{N}_G(\phi(P),Q,e_Q)e_Q}\mathbf{Perm}$, and not ${}_{k\bar{N}_G(Q,e_Q)e_Q}\mathbf{Perm}$. A better approach might be to consider a representation of a biset 2-category, following [Bo-1996]. We do not know whether this idea has already been explored.

We now give a partial generalisation of the above ideas to more general slash functors. In so doing, we raise more questions than answers... Let G be a finite group and e be a block of the group G. Let $\mathcal{S}$ be a set of compatible fusion-stable endopermutation source triples of the group G with respect to the block

e. For any arrow $\phi : (Q, e_Q) \to (P, e_P)$ in the Brauer category $\mathbf{Br}(G, e)$, any triple (P, e_P, V) in the set $\mathcal{S}$ and any capped indecomposable direct summand W of the kQ-module $\mathrm{Res}_\phi V$, we suppose that the triple (Q, e_Q, W) lies in the set $\mathcal{S}$. For any e-subpair (P, e_P) of the group G, we let $\mathcal{S}_{(P,e_P)}$ be the set of triples of the form $(Q, e_Q, V(P))$, where (Q, e_Q, V) is an element of the set $\mathcal{S}$ such that $(P, e_P) \triangleleft (Q, e_Q)$, and $V(P)$ is a P-slashed module attached to the endopermutation kQ-module V. Then $\mathcal{S}_{(P,e_P)}$ is a set of compatible fusion-stable source triples of the group $N_G(P, e_P)$ with respect to the block e_P. For a subgroup H of G such that $C_G(P) \leqslant H \leqslant N_G(P, e_P)$, we denote by ${}_{k\bar{H}e_P}\mathbf{M}$ the Brauer-friendly subcategory of the category ${}_{k\bar{H}e_P}\mathbf{Mod}$ (where $\bar{H} = PH/P$) that admits, as objects, the direct sums of indecomposable $kPHe_P$-modules with source triples in $\mathcal{S}_{(P,e_P)}$.

If ϕ is a 1-arrow of the 2-category $\mathbf{Br}(G, e)$, the definition of a ϕ-slashed functor can easily be deduced from Definition 3.20 and the above remarks on the Brauer functor Br_ϕ. Here is what we can prove about such objects.

Lemma 3.29. *(i) For any normal 1-arrow $\phi : (P, e_P) \to (Q, e_Q)$ in $\mathbf{Br}(G, e)$, there exists a ϕ-slash functor*

$$\mathrm{Sl}_\phi : {}_{k\bar{N}_G(P,e_P)e_P}\mathbf{M} \;\to\; {}_{k\bar{N}_G(\phi(P),Q,e_Q)e_Q}\mathbf{M},$$

which is unique up to isomorphism, up to twisting by a linear character $\chi : N_G(\phi(P), Q, e_Q)/C_G(Q) \to k^\times$.

(ii) Let $\phi_1, \phi_2 : (P, e_P) \to (Q, e_Q)$ be two normal 1-arrows in the 2-category $\mathbf{T}(G, e)$ such that there exist a 2-arrow $\phi_1 \to \phi_2$ in $\mathbf{T}(G, e)$. Up to twisting one of Sl_{ϕ_1} and Sl_{ϕ_2} by a linear character $\chi : N_G(\phi(P), Q, e_Q)/C_G(Q) \to k^\times$, there exists an isomorphism of functors

$$\eta_{\phi_2,\phi_1} : \mathrm{Sl}_{\phi_1} \;\to\; \mathrm{Sl}_{\phi_2},$$

which is unique up to scalar multiplication

(iii) Let $\phi : (P, e_P) \to (Q, e_Q)$ and $\psi : (Q, e_Q) \to (R, e_R)$ be two normal 1-arrows in the 2-category $\mathbf{T}(G, e)$, such that the composition $\psi\phi : (P, e_P) \to (R, e_R)$ is again a normal 1-arrow. Up to twisting one of the functors $\mathrm{Res}^{\bar{N}_G(\psi(Q),R,e_R)}_{\bar{N}_G(\psi\phi(P),\psi(Q),R,e_R)} \mathrm{Sl}_\psi$ and $\mathrm{Res}^{\bar{N}_G(\psi\phi(P),R,e_R)}_{\bar{N}_G(\psi\phi(P),\psi(Q),R,e_R)} \mathrm{Sl}_{\psi\phi}$ by a linear character $\chi : N_G(\psi\phi(P), \phi(Q), R, e_R)/C_G(R) \to k^\times$, there exists, for object M

of the category ${}_{k\bar{N}_G(P,e_P)e_P}\mathbf{M}$*, an isomorphism of* $kN(\psi\phi(P), \psi(Q), R, e_R)$*-modules*

$$\theta_{\psi,\phi}(M) : \mathrm{Sl}_{\psi\phi}(M) \;\to\; \mathrm{Sl}_{\psi} \circ \mathrm{Sl}_{\phi}(M),$$

which is natural in M*. This natural isomorphism is unique up to scalar multiplication.*

Proof. This is straightforward from Theorem 3.21 and Lemma 3.16. □

It would certainly be interesting to answer the following question.

Question 3.30. *With the above notations, is it possible to choose simultaneously the slash functors* Sl_ϕ*, for every 1-arrow* ϕ *in* $\mathbf{T}(G, e)$*, in such a way that, for every relevant pairs of 1-arrows* ϕ_1, ϕ_2 *or* ϕ, ψ*, there exist natural isomorphisms* η_{ϕ_2,ϕ_1}*,* $\theta_{\psi,\phi}$ *as in Lemma 3.29 (ii) and (iii)?*

Is it moreover possible to choose simultaneously the natural isomorphisms η_{ϕ_2,ϕ_1}*,* $\theta_{\psi,\phi}$*, for every relevant pairs of 1-arrows, in such a way that we obtain commutative diagrams similar to those of Lemma 3.27 (i) and Lemma 3.28 (ii) and (iii)?*

Chapitre 4
Contrôle de la fusion forte et équivalences stables

Chapter 4
Strong fusion control and stable equivalences

The Z_p^*-theorem may be seen as a statement that connects an assumption about p-fusion control to a conclusion about a Morita equivalence between the principal blocks of two finite groups. For an odd prime p, no modular proof of the Z_p^*-theorem has been found yet, but Broué gave a modular proof of a weaker statement, which leads to a stable equivalence between principal blocks.

In this chapter, we generalise that statement to nonprincipal blocks. Let G be a finite group, and $H = C_G(P)$ be the centraliser of a p-subgroup P of G; let e be a block of the algebra $\mathcal{O}G$, and e_P be a block of the algebra $\mathcal{O}H$ such that $\bar{e}_P \operatorname{br}_P(e) \neq 0$. Assuming that a defect group of the block e is abelian or has a noncyclic center, and that the centraliser H controls the e-fusion in the group G in a very strong sense, we prove that there exists a stable equivalence of Morita type between the block algebras $\mathcal{O}Ge$ and $\mathcal{O}He_P$.

Let G be a finite group, and H be a subgroup of G. If the group G admits the factorisation

$$G \;=\; O_{p'}(G)\, H,$$

then the subgroup H controls the p-fusion in the group G (this will be proven as Fact 4.4). *A fortiori*, the subgroup H controls the e-fusion in the group G, for any block e of the algebra $\mathcal{O}G$. This results admits at least two partial converses, which we gather in the following statement.

Theorem (Frobenius, Glauberman, classification)**.** *Let G be a finite group, and H be a subgroup that controls the p-fusion in G. Suppose that either of the following two assumptions is satisfied.*

- *$H = P$ is a p-subgroup of G (**Frobenius normal p-complement theorem**).*
- *$H = C_G(P)$ is the centraliser of a p-subgroup P of G (**Z_p^*-theorem**).*

Then the group G admits the factorisation $G = O_{p'}(G)H$.

The two cases $H = P$ and $H = C_G(P)$ share another property. For both of them, it is quite easy to prove that the group G admits the factorisation $G = O_{p'}(G)H$ if, and only if, the restriction functor Res^G_H induces an equivalence of categories

$$\mathrm{Res}^G_H : {}_{\mathcal{O}Ge_0}\mathbf{Mod} \;\to\; {}_{\mathcal{O}Hf_0}\mathbf{Mod},$$

where e_0 and f_0 are the principal blocks of the group algebras $\mathcal{O}G$ and $\mathcal{O}H$, respectively (this will be proven as Fact 4.5 for $H = C_G(P)$). Thus the above theorem can be rephrased as a statement that connects an assumption about fusion control and a conclusion about modular representations.

Theorem. *Let G be a finite group, and H be a subgroup that controls the p-fusion in G. Suppose that either of the following two assumptions is satisfied.*

- *$H = P$ is a p-subgroup of G (**Frobenius normal p-complement theorem**).*
- *$H = C_G(P)$ is the centraliser of a p-subgroup P of G (**Z_p^*-theorem**).*

Then the $(\mathcal{O}G, \mathcal{O}H)$-bimodule $\mathrm{Res}^{G\times G}_{G\times H}\,\mathcal{O}Ge_0$, i.e., *the restriction functor* Res^G_H*, induces a Morita equivalence between the principal block algebras $\mathcal{O}Ge_0$ and $\mathcal{O}Hf_0$.*

The normal p-complement theorem was proven by Frobenius in 1905. In its second formulation, it admits the following generalisation to a non-principal block, which was proven in [BP-1980] for a block with abelian defect group, and in [Pu-1988b] in the general case.

Theorem (Puig)**.** *Let G be a finite group, and e be a block of the algebra $\mathcal{O}G$. Let D be a defect group of the block e. Suppose that the subgroup D controls the e-fusion in G,* i.e.*, the block e is "nilpotent". Then there exists a Morita equivalence between the principal block algebra $\mathcal{O}Ge$ and the group algebra $\mathcal{O}D$.*

The Z_p^*-theorem was proven by Glauberman [Gl-1966] for $p = 2$. The original assumption was not clearly about fusion, but it is easily checked that our formulation is equivalent to the original one (we prove it in Fact 4.3). Glauberman's so-called Z^*-theorem was an important tool in the classification of finite simple group, which was completed in the early 1980's. Then the classification made it possible to prove the Z_p^*-theorem for an odd prime p; a proof of the odd Z_p^*-theorem can be found in [Ar-1988, Theorem 1] or [GLS-1998, Remark 7.8.3].

As we have shown above, the Z_p^*-theorem is a statement that essentially belongs to the modular representation theory. For $p = 2$, Glauberman's proof relies on the basic tools of this theory, such as Brauer's three main theorems. Therefore, it is quite unsatisfying that the only known proof of the Z_p^*-theorem, for p odd, relies on the classification of finite simple groups. A lot of work has been done in the last thirty years in order to find a modular proof of the odd Z_p^*-theorem, and a good deal oresults have been proven along the way.

From now on, let p be an odd prime number. A usual method of proof, in the theory of finite groups, is induction. So we may forget about the classification of finite simple groups, consider a counter-example of minimal order to the Z_p^*-theorem, and try to reach a contradiction. Let the group G be such a minimal counter-example. Let P be a non-trivial p-subgroup of G such that the centraliser $H = C_G(P)$ controls the p-fusion in G. Then the group G admits a Sylow p-subgroup D such that

- $P \leqslant D \leqslant H$;
- $N_G(Q) = O_{p'}(C_G(Q))\, N_H(Q)$ for any non-trivial p-subgroup $Q \leqslant D$.

(this will be proven as Fact 4.6). By Alperin's fusion theorem, these two conditions strengthen the assumption that the centraliser $H = C_G(P)$ controls the p-fusion in the group G. As we have mentioned already, proving the Z_p^*-theorem amounts to proving a Morita equivalence between principal blocks. The following weaker statement is due to Broué.

Theorem (Broué). *Let G be a group, and let $H = C_G(P)$ be the centraliser of a p-subgroup P. Suppose that the group G admits a Sylow p-subgroup D such that*

- $P \leqslant D \leqslant H$;
- $N_G(Q) = O_{p'}(C_G(Q))\, N_H(Q)$ *for any non-trivial p-subgroup $Q \leqslant D$.*

Then the $(\mathcal{O}G, \mathcal{O}H)$-bimodule $\mathrm{Res}^{G\times G}_{G\times H}\, \mathcal{O}Ge_0$, i.e., *the restriction functor Res^G_H, induces a stable equivalence of Morita type between the principal block algebras $\mathcal{O}Ge_0$ and $\mathcal{O}Hf_0$.*

The idea of Broué's proof is that a p-permutation bimodule that induces Morita equivalences between the principal blocks of the "local" subgroups must induce a stable equivalence between the principal blocks at the "global" level. A precise statement may be found in [Rq-2001, Theorem 5.6]. This stable equivalence has a consequence in terms of ordinary characters, the Reynolds isometry, which Robinson used in [Rb-1984] to prove character-theoretic results that point towards a modular proof of the odd Z_p^*-theorem.

Then a series of articles generalised those results about the principal block to similar results about non-principal blocks. In [Rb-1986] and [KR-1986], Külshammer and Robinson obtained Morita equivalences between nonprincipal block algebras of the groups G and H, whenever $H = C_G(P)$ is the centraliser of a p-group and $G = O_{p'}(G)\, H$. In [Rb-2009], Robinson proved character-theoretic results on nonprincipal blocks, as a consequence of what seemed to be the shadow of a stable equivalence between nonprincipal blocks. This generalisation was suggestive of the main result of this thesis, which we can now state.

Theorem 4.1. *We assume that the prime p is odd. Let G be a group, and e be a block of the algebra $\mathcal{O}G$. Let $H = C_G(P)$ be the centraliser of a p-subgroup P of G, and e_P be a block of the algebra $\mathcal{O}H$ such that $\bar{e}_P\, \mathrm{br}_P(e) \neq 0$. Suppose that the group G admits a maximal e-subpair (D, e_D) such that*

- $(P, e_P) \leqslant (D, e_D)$ *and* $D \leqslant H$;
- $N_G(Q, e_Q) = O_{p'}(C_G(Q))\, N_H(Q, e_Q)$ *for any non-trivial e-subpair $(Q, e_Q) \leqslant (D, e_D)$.*

Suppose moreover that the defect group D is abelian, or that its center $Z(D)$ is noncyclic. Then there exists a stable equivalence of Morita type between the block algebras $\mathcal{O}Ge$ and $\mathcal{O}He_P$.

As in Broué's theorem, the assumptions of Theorem 4.1 strengthen the condition that the centraliser $C_G(P)$ controls the e-fusion in the group G. The restrictive assumption on the defect group D (*i.e.*, that it should be abelian or have a non-cyclic center) is here for technical reasons; we need it to apply a theorem on the gluing of endopermutation modules. We believe that this assumption is not really necessary, and that it should be possible to get rid of it in the future.

If Theorem 4.1 could be proven without any assumption on the defect group D, it would bring the following consequence in the context of a minimal counter-example to the odd Z_p^*-theorem.

Corollary 4.2 (of Theorem 4.1, if there were no assumption on the defect group). *We assume that the prime p is odd. Let G be a group, and let $H = C_G(P)$ be the centraliser of the p-subgroup P of G. Suppose that the group G admits a Sylow p-subgroup D such that*

- $P \leqslant D \leqslant H$;
- $N_G(Q) = O_{p'}(C_G(Q))\, N_H(Q)$ *for any non-trivial p-subgroup* $Q \leqslant D$.

Let f be the sum of the blocks e of the algebra $\mathcal{O}G$ such that $\mathrm{br}_P(e) \neq 0$. Then there exists a stable equivalence of Morita type between the algebra $\mathcal{O}Gf$ and the whole group algebra $\mathcal{O}H$.

Indeed, with the assumptions of Corollary 4.2, the centraliser $H = C_G(P)$ contains the normaliser $N_G(Q)$ of any p-subgroup Q of G that contains P. Then it follows from Brauer's first main theorem that the Brauer map br_P induces a one-to-one correspondence $e \leftrightarrow e_P$ between the set of blocks e of the algebra $\mathcal{O}G$ with a defect group that contains P, and the set of all blocks e_P of the algebra $\mathcal{O}H$.

Actually, it is not clear whether Theorem 4.1 could be useful in order to find a modular proof of the Z_p^*-theorem. Instead, it could be that Theorem 4.1 points towards a possible generalisation of the Z_p^*-theorem for nonprincipal blocks. Such a generalisation might be stated in terms of Morita or derived equivalences, like Puig's theorem on nilpotent blocks. But we have no precise statement to suggest.

Let us now describe the organisation of this chapter. The general idea of our proof of Theorem 4.1 is to study a family of local Morita equivalences attached to the non-trivial e-subpairs $(Q, e_Q) \leqslant (D, e_D)$, and glue them together to obtain a stable equivalence of Morita type between the blocks e and e_P. This plan of action was first used by Puig in [Pu-1991], in order to prove a stable equivalence relative to a block with abelian defect group.

For the reader's convenience, Section 4.1 is devoted to a few facts that we have been stating in this introduction, in relation with the Z_p^*-theorem. These results are well-known, but not all of them are easy to find in the literature.

With the notations and assumptions of Theorem 4.1, let $(Q, e_Q) \leqslant (D, e_D)$ be a non-trivial e-subpair. Section 4.2 is devoted to the "local" $N_G(Q, e_Q)/C_G(Q)$-equivariant Morita equivalence

$$kC_G(Q)\bar{e}_Q \ \sim \ kC_H(Q)\,\mathrm{br}_P(\bar{e}_Q),$$

which is essentially the main result of [KR-1986]. We prove that this Morita equivalence is induced by a Brauer-friendly $k(C_G(Q) \times C_H(Q))\Delta N_H(Q, e_Q)$-module M_Q. Our construction of the Morita module M_Q makes a quite unexpected use of the class sums of the elements of the p-group P. This shows again the importance of these class sums in relation with the odd Z_p^*-theorem, which was already apparent in [Rb-1986] and [Rb-2009]. The indecomposable module M_Q admits a vertex subpair $(\Delta R, f \otimes f^{\mathrm{o}})$, in the group $(C_G(Q) \times C_H(Q))\Delta N_H(Q, e_Q)$, that contains the normaliser subpair

$$N_{(\Delta D, e_D \otimes e_D^{\mathrm{o}})}(\Delta Q) = (\Delta N_D(Q), e_{N_D(Q)} \otimes e_{N_D(Q)}^{\mathrm{o}}).$$

We consider a source of the indecomposable module M_Q with respect to the vertex subpair $(\Delta R, f \otimes f^{\mathrm{o}})$, and we restrict it to a module V_Q over the p-group $\Delta N_D(Q)$. We denote by v_Q the class of the endopermutation $k\Delta N_D(Q)$-module V_Q in the Dade group $\mathcal{D}(\Delta N_D(Q))$

In Section 4.3, we get to the gluing work. More precisely, we check that the local construction of Section 4.1 leads to a well-defined and properly compatible family $(v_Q)_{1 \neq Q \leqslant D}$ of classes of endopermutation modules. Then we use a result of Puig [Pu-1991], when D is abelian, or its generalisation by Bouc and Thévenaz [BT-2008], when D has a non-cyclic center, to glue up the family $(v_Q)_{1\neq Q\leqslant D}$, and obtain a class v in the Dade group $\mathcal{D}(\Delta D)$.

Next, we use the correspondence of Theorem 3.24 to construct a Brauer-friendly $\mathcal{O}(G\times H)$-module M with vertex subpair $(\Delta D, e_D\otimes e_D^{\mathrm{o}})$, and a fusion-stable endopermutation source V that belongs to the class $v\in\mathcal{D}(\Delta D)$. We prove that the slash construction maps the global bimodule M, as expected, to the family of local bimodules $(M_Q)_{1\neq Q\leqslant D}$ of Section 4.2. This proof works by descending induction on the subpair (Q, e_Q), a method that we borrow from [Rq-0000], and which may have its own interest.

We finally obtain a Brauer-friendly bimodule M that induces local Morita equivalences. Then it follows from a result of Linckelmann [Li-2013] that the bimodule M induces a stable equivalence between the block algebras $\mathcal{O}Ge$ and $\mathcal{O}He_P$.

4.1 Classical facts related to the Z_p^*-theorem

Fact 4.3. *Let G be a finite group, and P be a p-subgroup of G. Let D be a Sylow p-subgroup of G that contains P. The centraliser $C_G(P)$ controls the p-fusion in the group G if, and only if, no proper G-conjugate of an element of the p-group P lies in the Sylow p-subgroup D.*

Proof. Assume that the centraliser $C_G(P)$ controls the p-fusion in the group G. Then the subgroup $C_G(P)$ contains a Sylow p-subgroup D' of G, which must in turn contain the central p-subgroup P of $C_G(P)$. By Sylow's theorem, there exists an element $g\in G$ such that ${}^gD=D'$, hence ${}^gP\leqslant D'$. By the fusion control assumption, there exists an element $h\in C_G(P)$ such that $h^{-1}g\in C_G(P)$, *i.e.*, the element g lies in $C_G(P)$. Thus the original Sylow p-subgroup D lies in $C_G(P)$. Let $x\in P$ and $g\in G$ be such that the conjugate ${}^gx$ lies in D, and apply the fusion control assumption to the p-subgroup $Q=\langle x\rangle$. It follows that ${}^gx={}^hx$ for some $h\in C_G(P)$, so that ${}^gx=x$. We have proven that no proper G-conjugate of an element of P lies in D.

Conversely, assume that no proper G-conjugate of an element of P lies in D. Then no element of P has a proper D-conjugate, *i.e.*, the Sylow p-subgroup

D is contained in the centraliser $C_G(P)$. Since the Sylow p-subgroups of $C_G(P)$ are all conjugate in $C_G(P)$, it follows that no proper G-conjugate of an element of P lies in $C_G(P)$. Let Q be a p-subgroup of G that contains P. By Sylow's theorem, there exists $g \in G$ such that ${}^gQ \leqslant D$. For any element $x \in P$, it follows that ${}^gx \in D$, so that ${}^gx = x$. Hence the element g centralises P, so that the p-subgroup $Q \leqslant {}^{g^{-1}}D$ lies in $C_G(P)$.

Let Q be a subgroup of D and g be an element of D such that ${}^gQ \leqslant D$. Then the p-groups Q and ${}^gQ$ both lie in $C_G(P)$ or, equivalently, the p-groups P and ${}^{g^{-1}}P$ both lie in $C_G(Q)$. By Sylow's theorem, there exists an element $c \in C_G(Q)$ such that the p-subgroups P and ${}^{cg^{-1}}P$ lie in the same Sylow p-subgroup R of $C_G(Q)$. We set $h = gc^{-1}$, so that $h^{-1} = cg^{-1}$. By the above arguments, we deduce that the p-subgroup R of G, which contains P, must lie in $C_G(P)$, and that the p-subgroup ${}^{h^{-1}}P \leqslant R \leqslant C_G(P)$ cannot contain a proper G-conjugate of an element of P. This forces the element h to lie in the centraliser $C_G(P)$. We have $h^{-1}g = c \in C_G(Q)$, so we have proven that the subgroup $C_G(P)$ controls the p-fusion in the group G. □

Fact 4.4. *Let G be a finite group, P be a p-subgroup of G, and S be a normal p'-subgroup of G such that $G = S\,C_G(P)$.*

(i) No proper G-conjugate of an element of P centralises P.

(ii) The subgroup P is central in any p-subgroup of G that contains it.

(iii) For any p-subgroup Q of the centraliser $C_G(P)$, we have

$$C_G(Q) = C_S(Q)C_G(PQ) \quad \textit{and} \quad N_G(Q) = C_S(Q)N_{C_G(P)}(Q).$$

(iv) The centraliser $C_G(P)$ controls the p-fusion in the group G.

Proof. The key result is that no proper conjugate of an element of P lies in a p-subgroup of G which contains P. Indeed, let $x \in P$ and $g \in G$ be such that ${}^gx \in Q$, with Q a p-subgroup of G which contains P. Write $g = sc$ with $s \in S$ and $c \in C_G(P)$. Then $[g, x] = ({}^gx)x^{-1} \in Q$ and $[g, x] = [s, x] = s({}^xs)^{-1} \in S$ since S is a normal subgroup of G. Thus $[g, x] \in S \cap Q = 1$, and ${}^gx = x$.

In particular, if x' is a proper conjugate of an element $x \in P$ and x' centralises P, then x' lies in the p-subgroup $Q = P\langle x'\rangle$, a contradiction which proves (i). If Q is a p-subgroup of G containing P, then we have ${}^gx \in Q$ for any $g \in Q$ and any $x \in P$, so ${}^gx = x$ and (ii) is proven.

Next, let Q be any p-subgroup of the centraliser $C_G(P)$, and choose $g \in G$ such that ${}^gQ \leqslant C_G(P)$. There exist $s \in S$ and $c \in C_G(P)$ such that $g = c^{-1}s$. Then ${}^{cg}Q = {}^sQ$ is contained in $C_G(P)$ and in SQ, hence in the intersection $C_G(P) \cap SQ = C_S(P)Q$. So ${}^{cg}Q$ is a Sylow p-subgroup of $C_S(P)Q$ and it is conjugate to Q in $C_S(P)Q$: there exists $z \in C_S(P)$ such that ${}^{zcg}Q = Q$. By setting $h = (zc)^{-1} \in C_G(P)$, we obtain $h^{-1}g = zcg \in N_S(Q) = C_S(Q)$. This proves (iii).

The factorisation $G = SC_G(P)$, with S a normal p'-subgroup of G, implies that the prime p does not divide the index $[G : C_G(P)]$. Thus the subgroup $C_G(P)$ contains a Sylow p-subgroup D of G. Then (iv) follows from the above proof of (iii). Alternatively, (iv) can be deduced from (i) through Fact 4.3. □

Fact 4.5. *Let G be a finite group, and $H = C_G(P)$ be the centraliser of an abelian p-subgroup P of G. Let e_0 be the principal block of the group algebra $\mathcal{O}G$, and f_0 be the principal block of the group algebra $\mathcal{O}H$. The group G admits the factorisation $G = O_{p'}(G)\, H$ if, and only if, the restriction $\mathrm{Res}^{G\times G}_{G\times H} \mathcal{O}Ge_0$ is an $(\mathcal{O}Ge_0, \mathcal{O}Hf_0)$-bimodule that induces a Morita equivalence between the principal block algebras $\mathcal{O}Ge_0$ and $\mathcal{O}Hf_0$.*

Proof. We know from [Na-1998] that $O_{p'}(G)$ is the kernel of the group morphism $G \to (\mathcal{O}Ge_0)^\times$, $g \mapsto ge_0$.

Assume that the group G admits the factorisation $G = O_{p'}(G)H$. For any element $g \in G$, there exists $s \in O_{p'}(G)$ and $h \in C_G(P)$ such that $g = sh$; since $se_0 = e_0$, we have $ge_0 = he_0$. On the one hand, this implies $g(e_0f_0)g^{-1} = e_0(hf_0h^{-1}) = e_0f_0$ since $e_0 \in Z(\mathcal{O}G)$ and $f_0 \in Z(\mathcal{O}H)$. Moreover $\mathrm{br}_P(e_0f_0) = \mathrm{br}_P(e_0)\bar{f}_0 = \bar{f}_0 \neq 0$ by Brauer's third main theorem. Thus e_0f_0 is a non-zero central idempotent of the block algebra $\mathcal{O}Ge_0$, which means that $e_0f_0 = e_0$. On the other hand, the above computation implies $\mathcal{O}Ge_0 = \mathcal{O}He_0$. As a consequence, the map $\phi : \mathcal{O}Hf_0 \to \mathcal{O}Ge_0, x \mapsto xe_0$, is onto. Furthermore, the map $\mathrm{br}_P : (kG\bar{e}_0)^P \to kH\bar{f}_0$ is also onto. Thus the free $\mathcal{O}$-modules $\mathcal{O}Ge_0$ and $\mathcal{O}Hf_0$ must have the same $\mathcal{O}$-rank, and the surjective map ϕ must be an isomorphism. It follows that the restriction $\mathrm{Res}^{G\times G}_{G\times H} \mathcal{O}Ge_0$ is an $(\mathcal{O}Ge_0, \mathcal{O}Hf_0)$-bimodule that induces a Morita equivalence $\mathcal{O}Ge_0 \sim \mathcal{O}Hf_0$.

Conversely, assume that the restriction $\mathrm{Res}^{G\times G}_{G\times H} \mathcal{O}Ge_0$ is an $(\mathcal{O}Ge_0, \mathcal{O}Hf_0)$-

bimodule that induces a Morita equivalence $\mathcal{O}Ge_0 \sim \mathcal{O}Hf_0$. Then we have

$$\mathcal{O}Hf_0 \;\simeq\; \mathrm{End}_{\mathcal{O}Ge_0}(\mathcal{O}Ge_0).$$

In other words, the map $\phi : \mathcal{O}Hf_0 \to \mathcal{O}Ge_0, x \mapsto xe_0$, is an isomorphism of algebras. Since the subgroup Pf_0 is central in $(\mathcal{O}Hf_0)^\times$, it follows that the subgroup Pe_0 is central in $(\mathcal{O}Ge_0)^\times$, and that the subgroup P is central in G modulo the normal subgroup $O_{p'}(G)$. Then, for any $g \in G$, the p-subgroup ${}^gP$ lies in $O_{p'}(G)P$. Hence ${}^gP$ is a Sylow p-subgroup of $O_{p'}(G)P$, and there exists $s \in S$ such that the element $h = sg$ lies in the normaliser $N_G(P)$. Then we have $[h, P] \leqslant P \cap O_{p'}(G) = 1$, so the element h lies in the centraliser $C_G(P)$. We obtain $G = O_{p'}(G)C_G(P)$. □

We have proven Fact 4.5 only for P abelian, since a p-subgroup P such that $C_G(P)$ controls the p-fusion is necessarily abelian.

It should be mentioned here that the statement in Fact 4.5 is no longer true when the centraliser $C_G(P)$ is replaced with an arbitrary subgroup H of the group G. A counter-example may be derived from [Da-1977]. Let G be a finite group, D be a Sylow p-subgroup of G, and H be a normal subgroup of G such that $D \leqslant H$ and $G = H\,C_G(D)$. With the notations of Fact 4.5, Dade proves that the map $\phi : \mathcal{O}Hf_0 \to \mathcal{O}Ge_0, x \mapsto xe_0$, is an isomorphism. Thus the restriction $\mathrm{Res}^{G\times G}_{G\times H}\,\mathcal{O}Ge_0$ is an $(\mathcal{O}Ge_0, \mathcal{O}Hf_0)$-bimodule that induces a Morita equivalence between the principal block algebras $\mathcal{O}Ge_0$ and $\mathcal{O}Hf_0$. Now we choose $p = 3$ and we consider the symmetric group $G = S_5$, and the alternating subgroup $H = A_5$ of G. The group G admits a Sylow 3-subgroup D of order 3 such that $D \leqslant H$ and $G = H.C_G(D)$, so the above argument applies. But $O_{3'}(G)$ is trivial, so $G \neq O_{3'}(G)\,H$.

Fact 4.6. *Let the group G be a minimal counter-example to the odd Z_p^*-theorem: the group G does not satisfy the theorem, but any proper subquotient of the group G satisfies it. Let P be a non-trivial p-subgroup of G such that the centraliser $H = C_G(P)$ controls the p-fusion in G. Then the group G admits a Sylow p-subgroup D such that*

- *$P \leqslant D \leqslant H$;*
- *$N_G(Q) = O_{p'}(C_G(Q))\,N_H(Q)$ for any non-trivial p-subgroup $Q \leqslant D$.*

Proof. Let D be a Sylow p-subgroup of $C_G(P)$, hence of G by fusion control. Then D contains the central p-subgroup P of $C_G(P)$. Let Q be any non-trivial subgroup of the Sylow p-subgroup D. We have $Q \leqslant C_G(P)$, hence $P \leqslant C_G(Q)$. The centraliser $C_G(PQ)$ controls the p-fusion in $C_G(Q)$.

We first assume that the p-subgroup Q is central in G, and we set $\bar{G} = G/Q$. For any element g of G, write $\bar{g}$ for the direct image of g by the projection map $G \to \bar{G}$. The centraliser $C_{\bar{G}}(\bar{P})$ contains the projection $C_G(P)/Q$, so it controls the p-fusion in the quotient group $\bar{G}$. Since G is a minimal counter-example to the Z_p^*-theorem, we deduce that there exists a normal p'-subgroup $\bar{T}$ of $\bar{G}$ such that $\bar{G} = \bar{T}C_{\bar{G}}(P)$. We denote by T the preimage of $\bar{T}$ in the group G, so that $\bar{T} = T/Q$.

If an element $\bar{g} \in \bar{G}$ centralises the sugroup $\bar{P} = PQ/Q$ of $\bar{G}$, then the element $g \in G$ normalises the subgroup PQ of G. For any $x \in P$, the conjugate ${}^g x$ lies in the p-group PQ, which contains P. Since the centraliser $C_G(P)$ controls the p-fusion in G, this implies ${}^g x = x$ for any $x \in P$, so that the element g lies in the centraliser $C_G(P)$. It follows that $C_{\bar{G}}(P) = C_G(P)/Q$, so we obtain $G = TC_G(P)$. Moreover the p-group Q is a central Sylow p-subgroup of T, and we have $H^2(T/Q, Q) = 0$ because $T/Q = \bar{T}$ is a p'-group. So T is the direct product of the p-subgroup Q and a p'-subgroup S. In particular, the p'-subgroup $S = O_{p'}(T)$ is characteristic in T, hence normal in G. Since the subgroup Q centralises P, we obtain $G = SQC_G(P) = SC_G(P)$, *i.e.*, $N_G(Q) = O_{p'}(C_G(Q))\, N_H(Q)$.

We now assume that the centraliser $C_G(Q)$ is a proper subgroup of G. Since $C_G(Q)$ satisfies the assumptions of the Z_p^*-theorem and G is a minimal counter-example, we obtain $C_G(Q) \leqslant O_{p'}(C_G(Q))C_G(P)$. For any element g of the normaliser $N_G(Q)$, we have ${}^g Q = Q \leqslant D$ so there exists $h \in C_G(P)$ such that $h^{-1}g \in C_G(Q)$. In other words, we have $N_G(Q) \leqslant C_G(Q)C_G(P)$, which implies $N_G(Q) \leqslant O_{p'}(C_G(Q))C_G(P)$. This completes the proof. □

4.2 The local case

In this section, we work over the residue field k, except for the statement and proof of Lemma 4.12. We explore the Morita equivalence that was proven in [KR-1986]. In order to use it in the gluing procedure of the next section, we give an equivariant version of this Morita equivalence, and bring more details about the bimodule M that induces it.

Let us fix a few notations that will hold throughout this section. Let G be a finite group, and e be a block of the group G. Let (D, e_D) be a maximal e-subpair of the group G. Let H be a normal subgroup of G such that, for any e-subpair (Q, e_Q) of the group G such that $(Q, e_Q) \leqslant (D, e_D)$, the block e_Q of the algebra $kC_G(Q)$ is also a block of the algebra $kC_H(Q)$. Let P be a subgroup of the defect group D such that

$$G \;=\; O_{p'}(H)\; C_G(P).$$

We know from Fact 4.4 that this factorisation implies $N_G(D) \leqslant C_G(P)$. By Brauer's first main theorem, it follows that the idempotent $e_P = \mathrm{br}_P(e)$ is a block of the group $C_G(P)$, so that (P, e_P) is an e-subpair of the group G. The main result of this section is the following

Theorem 4.7. *With the above notations,*

(i) Let S be any p'-subgroup of H such that $S \triangleleft G$ and $G = SC_G(P)$; let b be any D-stable block of the group S such that $e_D\, \mathrm{br}_D(b) \neq 0$. The ΔD-algebra $kSb \otimes kC_S(P)\,\mathrm{br}_P(b)$ defines a class v of the Dade group $\mathcal{D}(\Delta D)$, which only depends on the maximal subpair (D, e_D) of the group G. Moreover, the class v is fusion-stable in the group $G \times C_G(P)$ with respect to the maximal subpair $(\Delta D, e_D \otimes e_D^{\circ})$.

(ii) Let V be a capped indecomposable endopermutation $k\Delta D$-module that belongs to the class v. There exists a unique indecomposable $k(H \times C_H(P))\Delta C_G(P)$-module M with source triple $(\Delta D, e_D \otimes e_D^{\circ}, V)$ such that the slashed module $M(\Delta P)$ is isomorphic to the $k(C_H(P) \times C_H(P))\Delta C_G(P)$-module $kC_H(P)e_P$.

(iii) The restriction of M to a $k(H \times C_H(P))$-module induces a Morita equivalence

$$kHe \;\sim\; kC_H(P)e_P.$$

We will reach the proof of this theorem through a series of lemmas. Firstly, we notice that the assumptions imply that the p-group P is abelian. We choose, once and for all, a normal p'-subgroup S of H such that $S \triangleleft G$ and $G = SC_G(P)$. For example, we could choose $S = O_{p'}(H)$. We know from Fact 4.4 that the centraliser $C_G(P)$ controls the p-fusion, and *a fortiori* the e-fusion, in the group G.

We then choose a block b of the p'-group S such that $eb \neq 0$, *i.e.*, the block e covers the block b (see for example [Da-1973] about covering blocks). We denote by G_b and H_b the stabilisers of b in the groups G and H respectively, and by $b' = \mathrm{Tr}_{G_b}^G(b)$ the sum of the G-conjugates of b. Then $eb' = e$ and $e \in kGb'$; moreover $e_b = eb$ is a block of the algebra kG_b, and the (kGe, kG_be_b)-bimodule kGe_b induces a Morita equivalence $kGe \sim kG_be_b$, as is proven for example in [Ha-2007].

If Q is a p-subgroup of G_b, we may consider the idempotent $b \in kS$ as a block of the p-nilpotent group SQ, with defect group Q. By Brauer's first Main Theorem, it follows that the idempotent $b_Q = \mathrm{br}_Q(b)$ is a block of the p-nilpotent group $N_{SQ}(Q) = C_S(Q)Q$ with defect group Q, *i.e.*, it is a block of the p'-group $C_S(Q)$. More precisely, the Brauer map br_Q induces a one-to-one correspondence between the set of Q-stable blocks of S and the set of all blocks of $C_S(Q)$. This is a special case of the Glauberman correspondence defined in [Gl-1968]. This correspondence commutes with the action of the normaliser $N_G(Q)$, so the stabiliser of the block b_Q in $N_G(Q)$ is $N_{G_b}(Q)$.

Lemma 4.8. *(i) We may choose the block b of the group S to be D-stable and such that, for any subpair $(Q, e_Q) \leqslant (D, e_D)$, the block e_Q of $kC_G(Q)$ covers the block $b_Q = \mathrm{br}_Q(b)$ of $kC_S(Q)$.*

(ii) A block b satisfying the condition in (i) is unique up to conjugation by an element of the centraliser $C_G(D)$.

(iii) If the block b satisfies the condition in (i), then the subgroup $C_{G_b}(P)$ controls the e-fusion in G with respect to the maximal subpair (D, e_D).

Proof. For any subgroup Q of D, we denote by e_Q the unique block of $\mathcal{O}C_G(Q)$ such that $(Q, e_Q) \leqslant (D, e_D)$.

The p'-subgroup $C_S(D)$ is normal in $C_G(D)$, so there exists a block c of the group $C_S(D)$ such that the block e_D of $C_G(D)$ covers c. Let b be the unique D-stable block of the p'-group S such that $\mathrm{br}_D(b) = c$, *i.e.*, $b_D = c$. By construction, the block e_D covers the block b_D. We use descending induction to generalise this to any p-subgroup of D. Let Q be a proper subgroup of D. We assume that, for any p-group R with $Q < R \leqslant D$, the block e_R of the group $C_G(R)$ covers the block b_R of the normal subgroup $C_S(R)$. We set $R = N_D(Q)$. By assumption, we have $e_R b_R \neq 0$. Since the p-group Q is normal in R, we also have $e_R\,\mathrm{br}_R(e_Q) = e_R$, which implies $\mathrm{br}_R(e_Q)b_R \neq 0$. Then $\mathrm{br}_R(e_Q b_Q) = \mathrm{br}_R(e_Q)b_R \neq 0$, so $e_Q b_Q \neq 0$ and the block e_Q covers the block b_Q. In particular, for $Q = 1$, we obtain that the block e covers the block b. This proves the statement in (i).

Since e_D is a block of the centraliser $C_G(D)$, a block c of $C_S(D)$ such that e_D covers c is unique up to conjugation in $C_G(D)$. Moreover, the correspondence $b \leftrightarrow c$ is one-to-one, so the statement in (ii) is proven.

Finally, let (Q, e_Q) be a subpair of (D, e_D) and let $g \in G$ be such that ${}^g(Q, e_Q) \leqslant (D, e_D)$. By Fact 4.4 (iv), the centraliser $C_G(P)$ controls the p-fusion in G, so we may suppose $g \in C_G(P)$. Then we obtain ${}^g(PQ, e_{PQ}) \leqslant (D, e_D)$, so we may suppose $P \leqslant Q$.

The inclusion ${}^g(Q, e_Q) \leqslant (D, e_D)$ implies ${}^g e_Q = e_{{}^gQ}$. So the block ${}^g e_Q$ of $C_G({}^gQ)$ covers the block $b_{{}^gQ}$ of $C_S({}^gQ)$, and the block e_Q of $C_G(Q)$ covers the block ${}^{g^{-1}}b_{{}^gQ}$ of $C_S(Q)$. As the blocks b_Q and ${}^{g^{-1}}b_{{}^gQ}$ of $C_S(Q)$ are covered by the same block e_Q of $C_G(Q)$, they must be conjugate in $C_G(Q)$: there exists $k \in C_G(Q)$ such that $b_Q = {}^{kg^{-1}}b_{{}^gQ}$. Then we get

$$\mathrm{br}_Q(b) \;=\; b_Q \;=\; {}^{kg^{-1}}b_{{}^gQ} \;=\; {}^{kg^{-1}}\mathrm{br}_{{}^gQ}(b) \;=\; \mathrm{br}_Q({}^{kg^{-1}}b).$$

As we have already mentionned, the correspondence $b \leftrightarrow \mathrm{br}_Q(b)$ is one-to-one, so we obtain $b = {}^{kg^{-1}}b$. Hence the element $h = gk^{-1}$ lies in G_b. Notice that g and k both centralise the p-group P by assumption, so we have $h \in C_{G_b}(P)$, and the statement in (iii) is proven. □

In [Rb-1986], with assumptions similar to ours, Robinson considered the central p-element $K_x e \in (kGe)^\times$, where K_x stands for the sum of all G-conjugate of the element x. In [Rb-2009], he made great use of the same central unit to deal with a minimal counter-example to the odd Z_p^*-theorem. The following lemmas will highlight once again the importance of the class sum K_x, which we use to define the bimodule M of Theorem 4.7.

In order to deal efficiently with this class sum, we will need more notations. We will consider the group G as a subgroup of the direct product $G \times P$, via the embedding $g \mapsto (g, 1)$. Since the p-group P is abelian, we can consider the

p-subgroup $P_1 = \{(x, x^{-1})\,;\; x \in P\}$ of $G \times P$. For an element $x \in P$, we will usually write $x_1 = (x, x^{-1}) \in P_1$; conversely, for an element $x_1 \in P_1$, we will write x for the unique element of P such that $x_1 = (x, x^{-1})$. We will see the group $G \times P$ as the semi-direct product GP_1. Notice that any subgroup of G that is normalised by P is also normalised by P_1. Thus we can consider the subgroups HP_1, SP_1, G_bP_1, *etc.* The group P_1 centralises the defect group D and stabilises the blocks e, e_D, b, *etc.*

We also need to clarify the notation K_x for an element $x \in P$. By definition, we have $K_x = \mathrm{Tr}_{C_G(x)}^G(x)$. We have supposed $G = SC_G(P)$ hence $G = SC_G(x)$. So, for any subgroup T of G that contains S, the natural map $T/C_T(x) \to G/C_G(x)$ is a bijection and $K_x = \mathrm{Tr}_{C_T(x)}^T(x)$. We have in particular $K_x = \mathrm{Tr}_{C_S(x)}^S(x)$. Since the subgroup S is normal in G, it follows that the class sum K_x lies in the subset $kSx = xkS$ of the algebra kG, and that the element $xK_{x^{-1}}$ lies in kS.

The following lemma extends the usual structures of interior algebras of kSb, kHb', and kHe.

Lemma 4.9. *(i) The maps $\iota_b : SP_1 \to (kSb)^\times$ and $\gamma_b : G_bP_1 \to \mathrm{Aut}_{\mathbf{Alg}}(kSb)$ defined by*

$$\iota_b(sx_1) = sxK_{x^{-1}}b \qquad ; \qquad \gamma_b(gx_1)(a) = (gx)^{-1}a(gx)$$

for $s \in S$, $g \in G_b$, $x_1 \in P_1$, $a \in kSb$, make kSb an SP_1-interior G_bP_1-algebra.

(ii) The maps $\iota_{b'} : HP_1 \to (kHb')^\times$ and $\gamma_{b'} : GP_1 \to \mathrm{Aut}_{\mathbf{Alg}}(kHb')$ defined by

$$\iota_{b'}(hx_1) = hxK_{x^{-1}}b' \qquad ; \qquad \gamma_{b'}(gx_1)(a) = (gx)^{-1}a(gx)$$

for $h \in H$, $g \in G$, $x_1 \in P_1$, $a \in kHb'$, make kHb' an HP_1-interior GP_1-algebra. Moreover, there is a natural isomorphism of HP_1-interior GP_1-algebras

$$kHb' \;\to\; \mathrm{Ind}_{(SP_1 \times SP_1)\Delta G_b}^{(HP_1 \times HP_1)\Delta G} kSb.$$

(iii) The maps $\iota_e : HP_1 \to (kHe)^\times$ and $\gamma_e : GP_1 \to \mathrm{Aut}_{\mathbf{Alg}}(kHe)$ defined by

$$\iota_e(hx_1) = hxK_{x^{-1}}e \qquad ; \qquad \gamma_e(gx_1)(a) = (gx)^{-1}a(gx)$$

for $h \in H$, $g \in G$, $x_1 \in P_1$, $a \in kHe$, make kHe an HP_1-interior GP_1-algebra.

Proof. We consider the idempotents b and $b_P = \mathrm{br}_P(b)$ as respective blocks of the p-nilpotent groups SP and $C_S(P)P$, both with defect group P. It is well known that the block algebras $kSPb$ and $kC_S(P)Pb_P$ are both Morita equivalent to kP, so the centers $Z(kSPb)$ and $Z(kC_S(P)Pb_P)$ are both isomorphic to $Z(kP) = kP$; in particular, they have the same dimension. The Brauer map br_P induces an algebra morphism $\beta : Z(kSPb) \to Z(kC_S(P)Pb_P)$. We have $kPb_P \subseteq Z(kC_S(P)Pb_P)$. Moreover the natural map $kC_S(P) \otimes kP \to kC_S(P)P$ is an isomorphism, so $\dim_k(kPb_P) = |P| = \dim_k Z(kC_S(P)Pb_P)$. Hence we have $Z(kC_S(P)Pb_P) = kPb_P$. For an element x in P, we remember from Fact 4.4 (i) that no proper conjugate of x lies in $C_G(P)$; so $\beta(K_x b) = \mathrm{br}_P(K_x)\,\mathrm{br}_P(b) = xb_P$. This proves that the morphism β is onto. Since its domain and codomain have the same dimension over k, β is an isomorphism. Thus the element $K_x b = \beta^{-1}(xb_P)$ is invertible in $Z(kSPb)$ and the map $x \mapsto K_x b$ is a group morphism $P \to Z(kSPb)^\times$. Since the group P is abelian, it follows that the map $\iota_b : SP_1 \to (kSb)^\times$ of (i) is indeed well-defined and a group morphism. The rest of the statement in (i) is straightforward.

Furthermore, the algebra kSb' is the direct product of the kSc where c runs over the set of G-conjugates of b, and $xK_{x^{-1}}b' = \sum_c xK_{x^{-1}}c$ for any $x \in P$. So $xK_{x^{-1}}b'$ is invertible in kSb' and the map $x_1 \mapsto xK_{x^{-1}}b'$ is a group morphism $P_1 \to (kSb')^\times$, which extends to the group morphism $\iota_{b'} : HP_1 \to (kHb')^\times$ of (ii). Notice that cutting off the central idempotent e cannot harm, so (iii) follows immediately from the first part of (ii).

We move on to the second part of (ii). On the one hand, we have from Section 1.2 an isomorphism of H-interior G-algebras

$$kHb' \to \mathrm{Ind}_{(S\times S)\Delta G_b}^{(H\times H)\Delta G} kSb.$$

On the other hand, the natural map $G/S \to GP_1/SP_1$ is a bijection so the natural map

$$\mathrm{Ind}_{(S\times S)\Delta G_b}^{(H\times H)\Delta G} kSb \to \mathrm{Ind}_{(SP_1\times SP_1)\Delta G_b}^{(HP_1\times HP_1)\Delta G} kSb$$

is an isomorphism of H-interior G-algebras. As a consequence, we obtain a natural isomorphism of H-interior G-algebras

$$\phi : kHb' \;\to\; \mathrm{Ind}_{(SP_1\times SP_1)\Delta G_b}^{(HP_1\times HP_1)\Delta G} kSb.$$

Since $G = SC_G(P)$, we have $b' = \mathrm{Tr}_{C_{G_b}(P)}^{C_G(P)} b$, hence $\iota_{b'}(x_1) = \mathrm{Tr}_{C_{G_b}(P)}^{C_G(P)}(\iota_b(x_1))$ for any $x_1 \in P_1$. Thus ϕ is also an isomorphism of P_1-interior algebras. This completes the proof of (ii). $\square$

We now consider the HP_1-interior GP_1-algebra kHe as a $k(HP_1 \times HP_1)\Delta G$-module.

Lemma 4.10. *(i) The restriction* $\mathrm{Res}_{(P_1\times P_1)\Delta D}^{(SP_1\times SP_1)\Delta G_b} kSb$ *is a capped endopermutation* $k(P_1 \times P_1)\Delta D$*-module, which is fusion-stable in the group* $GP_1 \times GP_1$ *with respect to the* $e \otimes e^{\mathrm{o}}$*-subpair* $((P_1 \times P_1)\Delta D, e_D \otimes e_D^{\mathrm{o}})$*. We denote by* w *the corresponding class in the Dade group* $\mathcal{D}((P_1 \times P_1)\Delta D)$*.*

(ii) The indecomposable $k(H_bP_1\times H_bP_1)\Delta G_b$*-module* kH_be_b *is an endo-*p*-permutation module with vertex subpair* $((P_1 \times P_1)\Delta D, e_Db_D \otimes e_D^{\mathrm{o}}b_D^{\mathrm{o}})$ *and a source that belongs to the class* w*.*

(iii) The indecomposable $k(HP_1\times HP_1)\Delta G$*-module* kHe *is Brauer-friendly with vertex subpair* $((P_1 \times P_1)\Delta D, e_D \otimes e_D^{\mathrm{o}})$ *and a source that belongs to the class* w*.*

Proof. The field k is algebraically closed and the group S is a p'-group, so the block algebras kSb is a matrix algebra. It follows that the structure map of the (kSb, kSb)-bimodule kSb is an isomorphism of $(SP_1 \times SP_1)\Delta G_b$-algebras

$$kSb \otimes kSb^{\mathrm{o}} \simeq \mathrm{End}_k(kSb).$$

In particular, this proves that kSb is an endo-p-permutation $k(SP_1 \times SP_1)\Delta G_b$-module. We temporarily set $A = (SP_1 \times SP_1)\Delta G_b$. For any element (g, h) of the group $(H_bP_1 \times H_bP_1)\Delta G_b$, let R be a Sylow p-subgroup of the intersection $A \cap {}^{(g,h)}A$. The $k(S \times S)R$-module $\mathrm{Res}_{(S\times S)R}^{A} kSb$ is simple and belongs to the block $b \otimes b^{\mathrm{o}}$ of the p-nilpotent group $(S \times S)R$. Since the pair (g, h) stabilises the block $b \otimes b^{\mathrm{o}}$, the $k(S \times S)R$-module $\mathrm{Res}_{(S\times S)R}^{(g,h)A} g(kSb)h^{-1}$ is still simple and belongs to the same block $b \otimes b^{\mathrm{o}}$. It is well-known that a block of a p-nilpotent group contains only one isomorphism class of simple modules (see for example [BP-1980]). Thus there is an isomorphism of $k(S \times S)R$-modules

$$\mathrm{Res}_{(S\times S)R}^{A} kSb \simeq \mathrm{Res}_{(S\times S)R}^{(g,h)A} g(kSb)h^{-1}.$$

It follows that the restrictions $\mathrm{Res}^{A}_{A\cap {}^{(g,h)}A}\, kSb$ and $\mathrm{Res}^{{}^{(g,h)}A}_{A\cap {}^{(g,h)}A}\, g(kSb)h^{-1}$ are compatible endo-p-permutation $k(A\cap {}^{(g,h)}A)$-modules. By Urfer's criterion [Ur-2006, Proposition 2.10] for the induction of endo-p-permutation modules, we deduce that the $k(H_bP_1 \times H_bP_1)\Delta G_b$-module

$$kH_bb \;\simeq\; \mathrm{Ind}^{(H_bP_1\times H_bP_1)\Delta G_b}_{(SP_1\times SP_1)\Delta G_b}\, kSb$$

is an endo-p-permutation module. Then its direct summand kH_be_b is also an endo-p-permutation $k(H_bP_1 \times H_bP_1)\Delta G_b$-module. We now determine a vertex subpair of this indecomposable module. The commutation of induction and the Brauer functor brings an isomorphism of $(C_{H_b}(P)P_1 \times C_{H_b}(P)P_1)\Delta C_{G_b}(P)$-interior algebras

$$\mathrm{Br}_{P_1\times P_1}(\mathrm{End}_k(kH_bb)) \;\to\; \mathrm{Ind}^{(C_{H_b}(P)P_1\times C_{H_b}(P)P_1)\Delta C_{G_b}(P)}_{(C_S(P)P_1\times C_S(P)P_1)\Delta C_{G_b}(P)}\, \mathrm{Br}_{P_1\times P_1}(\mathrm{End}_k(kSb)),$$

where the induction functor must be read as the induction of interior algebras. The isomorphism $\mathrm{End}_k(kSb) \simeq kSb \otimes kSb^{\mathrm{o}}$ brings an isomorphism of $(C_S(P)P_1 \times C_S(P)P_1)\Delta C_{G_b}(P)$-interior algebras

$$\mathrm{Br}_{P_1\times P_1}(\mathrm{End}_k(kSb)) \;\simeq\; kC_S(P)b_P \otimes kC_S(P)b_P^{\mathrm{o}} \;\simeq\; \mathrm{End}_k(kC_S(P)b_P).$$

With Lemma 4.9 (ii), this brings an isomorphism of $(C_{H_b}(P)P_1\times C_{H_b}(P)P_1)\Delta C_{G_b}(P)$-interior algebras

$$\mathrm{Br}_{P_1\times P_1}(\mathrm{End}_k(kH_bb)) \;\simeq\; \mathrm{End}_k(kC_{H_b}(P)b_P).$$

It follows that the slashed module $kH_be_b(P_1\times P_1)$ is isomorphic to $kC_{H_b}(P)\,\mathrm{br}_P(e_b)$. So a vertex of the indecomposable $k(H_bP_1 \times H_bP_1)\Delta G_b$-module kH_be_b contains the p-group $P_1 \times P_1$. Since $kC_{H_b}(P)\,\mathrm{br}_P(e_b)$ is a p-permutation $k(C_{H_b}(P) \times C_{H_b}(P))\Delta C_{G_b}(P)$-module, the slash construction may coincide with the Brauer functor from this point on. The images of the block algebra $kC_{H_b}(P)\,\mathrm{br}_P(e_b)$ by Brauer functors are well-known, so we can use the transitivity of the slash construction for endo-p-permutation modules, and conclude that a vertex subpair of $kC_{H_b}(P)\,\mathrm{br}_P(e_b)$ is $((P_1 \times P_1)\Delta D, e_Db_D \otimes e_D^{\mathrm{o}}b_D^{\mathrm{o}})$.

We denote by W a source of the indecomposable $k(H_bP_1\times H_bP_1)\Delta G_b$-module kH_be_b with respect to the above vertex subpair. Since kH_be_b is an endo-p-permutation module, the source W is an endopermutation module, and it is fusion-stable in the group $(H_bP_1 \times H_bP_1)\Delta G_b$ with respect to the p-subgroup $(P_1 \times P_1)\Delta D$. By Lemma 4.8 (iii), the subgroup $(H_bP_1 \times H_bP_1)\Delta G_b$ controls the $e \otimes e^{\mathrm{o}}$-fusion in the group $(HP_1 \times HP_1)\Delta G$ with respect to a maximal subpair that contains the subpair $((P_1 \times P_1)\Delta D, e_D \otimes e_D^{\mathrm{o}})$. Thus the source W is fusion-stable in the group $(HP_1 \times HP_1)\Delta G$ with respect to the $e \otimes e^{\mathrm{o}}$-subpair $((P_1 \times P_1)\Delta D, e_D \otimes e_D^{\mathrm{o}})$.

As a consequence of the above argument and of the vertex-preserving Morita equivalence of [Ha-2007, Theorem 1.6], the induced module

$$kHe \;\simeq\; \mathrm{Ind}_{(H_bP_1 \times H_bP_1)\Delta G_b}^{(HP_1 \times HP_1)\Delta G}\; kH_be_b$$

is an indecomposable Brauer-friendly $k(HP_1 \times HP_1)\Delta G$-module with vertex subpair $((P_1 \times P_1)\Delta D, e_D \otimes e_D^{\mathrm{o}})$ and source W.

Finally, since kH_bb is an endo-p-permutation $k(H_bP_1 \times H_bP_1)\Delta G_b$-module induced from the $k(SP_1 \times SP_1)\Delta G_b$-module kSb, the source W is isomorphic to any capped indecomposable direct summand of the restriction $\mathrm{Res}_{(P_1 \times P_1)\Delta D}^{(SP_1 \times SP_1)\Delta G_b}\; kSb$. This completes the proof. □

Let $kHe(1 \times P_1)$ be a $(1 \times P_1, e \otimes e_P^{\mathrm{o}})$-slashed module attached to the Brauer-friendly $k(HP_1 \times HP_1)\Delta G$-module kGe. From now on, we will restrict this slashed module to a $k(H \times C_H(P))\Delta C_G(P)$-module.

Lemma 4.11. *(i) The $k(H \times C_H(P))\Delta C_G(P)$-module $M = kHe(1 \times P_1)$ induces a G/H-equivariant Morita equivalence $kHe \sim kC_H(P)e_P$.*

(ii) The indecomposable $k(H \times C_H(P))\Delta C_G(P)$-module M is Brauer-friendly with vertex subpair $(\Delta D, e_D \otimes e_D^{\mathrm{o}})$.

(iii) Let V be the source of M with respect to the vertex subpair $(\Delta D, e_D \otimes e_D^{\mathrm{o}})$. We identify V with a kD-module through the natural isomorphism $D \simeq \Delta D$. In the Dade group $\mathcal{D}(D)$, we have the equality

$$[V] \;=\; u - u(P),$$

where $u \in \mathcal{D}(D)$ is the class defined by the Dade D-algebra kSb, and $u(P) \in \mathcal{D}(D)$ is the class defined by the Brauer quotient $\mathrm{Br}_P(kSb) \simeq kC_S(P)b_P$.

Proof. We apply the slash construction to the $k(SP_1 \times SP_1)\Delta G_b$-module kSb and the $k(HP_1 \times HP_1)\Delta G$-module kHb' to define a $k(S \times C_S(P))\Delta G_b$-module $L = kSb(1 \times P_1)$ and a $k(H \times C_H(P))\Delta C_G(P)$-module $L' = kHb'(1 \times P_1)$. We know from Lemma 4.9 (ii) that there is an isomorphism of $k(HP_1 \times HP_1)\Delta G$-modules $kHb' \simeq \mathrm{Ind}_{(SP_1\times SP_1)\Delta G_b}^{(HP_1\times HP_1)\Delta G} kSb$. Moreover $G = SC_G(P)$, so the commutation of the induction and the Brauer functor brings an isomorphism of $(HP_1 \times C_H(P)P_1)\Delta C_G(P)$-interior algebras

$$\mathrm{Ind}_{(SP_1\times C_S(P)P_1)\Delta C_{G_b}(P)}^{(HP_1\times C_H(P)P_1)\Delta C_G(P)} \circ \mathrm{Br}_{1\times P_1} \mathrm{End}_k(kSb) \to \mathrm{Br}_{1\times P_1} \circ \mathrm{Ind}_{(SP_1\times SP_1)\Delta G_b}^{(HP_1\times HP_1)\Delta G} \mathrm{End}_k(kSb).$$

Notice that the p-subgroup P_1 can be omitted from the induction functors without changing the result. Thus we have an isomorphism of $k(H\times C_H(P))\Delta C_G(P)$-modules

$$\mathrm{Ind}_{(S\times C_S(P))\Delta C_{G_b}(P)}^{(H\times C_H(P))\Delta C_G(P)} L \;\simeq\; L'.$$

Then we look closer at the definition of L. Since kSb is a matrix algebra, the structure map of the (kSb, kSb)-bimodule kSb is an isomorphism of $(S \times S)$-interior algebras $kSb \otimes (kSb)^{\mathrm{op}} \to \mathrm{End}_k(kSb)$. Applying the Brauer functor $\mathrm{Br}_{1\times P_1}$ turns this into an isomorphism of $(S \times C_S(P))$-interior algebras $kSb \otimes (kC_S(P)b_P)^{\mathrm{op}} \to \mathrm{End}_k(L)$. So we have an isomorphism of $k(S\times C_S(P))$-modules $L \simeq X \otimes Y^*$, where X is a simple module for the matrix algebra kSb and Y^* is the k-dual of a simple module for the matrix algebra $kC_S(P)b_P$.

We deduce that the $k(S \times C_S(P))$-module L induces a Morita equivalence $kSb \sim kC_S(P)b_P$. By [Ma-1996, Theorem 3.4], it follows that the induced module L' induces a Morita equivalence $kHb' \sim kC_H(P)b'_P$. Finally, it is clear that $M \simeq eL'e_P$. Since the module $eL'e_P$ is non-zero, it induces a Morita equivalence $kHe \sim kC_H(P)e_P$. This proves (i).

The indecomposable $k(HP_1 \times HP_1)\Delta G$-module kHe is Brauer-friendly with vertex subpair $((P_1\times P_1)\Delta D, e_D\otimes e_D^{\mathrm{o}})$. On the one hand, Theorem 3.15 (i) shows that the slashed module M is Brauer-friendly. On the other hand, Lemma 3.16 shows that there is an isomorphism

$$M(R, f) \;\simeq\; kHe((1 \times P_1)R, \mathrm{br}_{1\times P_1}(f))$$

for any subpair (R, f) in the group $(H \times C_H(P))\Delta C_G(P)$ such that $(R, f) \leqslant (D \times D, e_D \otimes e_D^{\mathrm{o}})$. It follows that $M(R, f)$ is non-zero if, and only if, the subpair

(R, f) is contained in $(\Delta D, e_D \otimes e_D^{\circ})$ up to conjugation. Thus $(\Delta D, e_D \otimes e_D^{\circ})$ is a vertex subpair of the indecomposable $k(H \times C_H(P))\Delta C_G(P)$-module M. This proves (ii).

Let V be the source of M with respect to the above vertex subpair. It follows from Theorem 3.15 (iii) that the endopermutation $k\Delta D$-module V is compatible with the slashed module $\mathrm{Res}^{(P_1 \times P_1)\Delta D}_{\Delta D} W(1 \times P_1)$. Moreover, we know from Lemma 4.10 that W is a capped indecomposable direct summand of the $k\Delta D$-module kSb. We have $\mathrm{End}_k(kSb) \simeq kSb \otimes kSb^{\circ}$, so $\mathrm{Br}_{1 \times P_1}(\mathrm{End}_k(kSb)) \simeq kSb \otimes kC_S(P)b_P^{\circ}$. Thus the $k\Delta D$-module V is isomorphic to a direct summand of a simple module for the matrix algebra $kSb \otimes kC_S(P)b_P^{\circ}$. This proves (iii). □

We have now proven most of Theorem 4.7, except a part of (ii). We have obtained an indecomposable $k(H \times C_H(P))\Delta C_G(P)$-module M with source triple $(\Delta D, e_D \otimes e_D^{\circ}, V)$. The fusion-stable endopermutation $k\Delta D$-module V belongs to the class v of Theorem 4.7 (i). Moreover we have $M = kHe(1 \times P_1)$ and kHe is a Brauer-friendly $k(HP_1 \times HP_1)\Delta G$-module. By Lemma 3.16, it follows that the slashed module $M(\Delta P)$ is isomorphic to $kHe(\Delta P)(1 \times P_1)$. It is easily checked that the $k(C_H(P)P_1 \times C_H(P)P_1)\Delta C_G(P)$-module $kHe(\Delta P)$ is isomorphic to $kC_H(P)e_P$, with a trivial action of $P_1 \times P_1$. Thus we have an isomorphism of $k(C_H(P) \times C_H(P))\Delta C_G(P)$-modules $M(\Delta P) \simeq kC_H(P)e_P$.

We conclude this section with a statement on the unicity of the bimodule M that completes the proof of Theorem 4.7 (ii). Actually, we prove a more general statement, which will also be used in the next section. For this lemma, we do not assume that $\mathcal{O} = k$.

Lemma 4.12. *Let G be a finite group, and e be a block of the group G. Let H be a normal subgroup of G such that, for any e-subpair (Q, e_Q) of the group G, the block e_Q of the algebra $kC_G(Q)$ is also a block of the algebra $kC_H(Q)$. Assume that the group G admits an e-subpair (P, e_P) and a Sylow e-subpair (D, e_D) such that*

- *$(P, e_P) \leqslant (D, e_D)$ and $D \leqslant C_G(P)$;*
- *$N_G(Q, e_Q) = C_G(Q)\, C_{N_G(Q,e_Q)}(P)$ for any e-subpair $(Q, e_Q) \leqslant (D, e_D)$.*

Let V be a capped indecomposable endopermutation $\mathcal{O}D$-module that is fusion-stable in the group G with respect to the subpair (D, e_D). Assume moreover *that the slashed module $V(P)$ is isomorphic to the trivial kD-module k. We identify V to an $\mathcal{O}\Delta D$-module through the natural isomorphism $D \simeq \Delta D$. Then, up to isomorphism, there is a unique indecomposable $\mathcal{O}(H \times C_H(P))\Delta C_G(P)$-module M with source triple $(\Delta D, e_D \otimes e_D^{\circ}, V)$ such that the slashed module $M(\Delta P, e_P \otimes e_P^{\circ})$ admits the $k(C_H(P) \times C_H(P))\Delta C_G(P)$-module $kC_H(P)\bar{e}_P$ as a direct summand.*

Proof. Let M be an indecomposable $\mathcal{O}(H \times C_H(P))\Delta C_G(P)$-module with source triple $(\Delta D, e_D \otimes e_D^{\circ}, V)$. Since the source V is fusion-stable, the module M is Brauer-friendly, so we know from Theorem 3.15 (i) that M admits a slashed module $M(\Delta P, e_P \otimes e_P^{\circ})$.

For the sake of shortness, whenever (Q, e_Q) is a subpair of (D, e_D), we write $N(Q, e_Q)$ for the group $(C_H(Q) \times C_H(Q))\Delta N_G(Q, e_Q)$ and $M(Q, e_Q)$ for the slashed module $M(\Delta Q, e_Q \otimes e_Q^{\circ})$. Let the $\mathcal{O}N(D, e_D)$-module M' be a Green correspondent of M with respect to the vertex subpair $(\Delta D, e_D \otimes e_D^{\circ})$. The $kN(D, e_D)$-module $kC_H(D)\bar{e}_D$ is a Green correspondent of the indecomposable $kN(P, e_P)$-module $kC_H(P)e_P$ with respect to the vertex subpair $(\Delta D, e_D \otimes e_D^{\circ})$. Then we deduce from Theorem 3.15 (iii) that the slashed module $M'(P, e_P)$ admits the $kN(D, e_D)$-module $kC_H(D)e_D$ as a direct summand if, and only if, the slashed module $M(P, e_P)$ admits the k-module $kC_H(P)e_P$ as a direct summand.

We consider the $\mathcal{O}N(D, e_D)$-module $L' = e_D(\mathrm{Ind}_{\Delta D}^{N(D,e_D)} V)e_D$, which is an endo-$p$-permutation module since the source V is $N(D, e_D)$-stable. The p-subgroup ΔP is central in $N(D, e_D)$, and the slashed module $V(P)$ is isomorphic to k. Thus the commutation of induction and the Brauer functor implies that there is an isomorphism of $kN(D, e_D)$-modules

$$L'(\Delta P) \simeq \bar{e}_D(\mathrm{Ind}_{\Delta D}^{N(D,e_D)} k)\bar{e}_D.$$

Let X be an indecomposable $kN(D, e_D)$-module with source triple $(\Delta D, e_D \otimes e_D^{\circ}, k)$, where k is the trivial $k\Delta D$-module. Then X is isomorphic to a direct summand of the $kN(D, e_D)$-module $\bar{e}_D(\mathrm{Ind}_{\Delta D}^{N(D,e_D)} k)\bar{e}_D$, hence of the slashed module $L'(P, e_P)$. As a consequence, there exists a primitive idempotent j in the algebra $\mathrm{End}_{kN(D,e_D)}(L'(P, e_P))$ such that $X \simeq jL'(P, e_P)$, and this idempotent is unique up to conjugation in the algebra $\mathrm{End}_{kN(D,e_D)}(L'(P, e_P))$. Since L' is an endo-p-permutation $kN(D, e_D)$-module, the Brauer map induces an epimorphism

$$\beta : \mathrm{End}_{\mathcal{O}N(D,e_D)}(L') \;\to\; \mathrm{End}_{kN(D,e_D)}(L'(P,e_P)).$$

Thus, by the lifting theorem [Th-1995, Theorem 3.2], there exists a primitive idempotent j' in the algebra $\mathrm{End}_{\mathcal{O}N(D,e_D)}(L')$ such that $\beta(j') = j$, and this idempotent is unique up to conjugation in the algebra $\mathrm{End}_{kN(D,e_D)}(L')$. Conversely, if X' is an indecomposable $\mathcal{O}N(D,e_D)$-module with source triple $(\Delta D, e_D \otimes e_D^{\circ}, V)$, then there exists a primitive idempotent i in the algebra $\mathrm{End}_{\mathcal{O}N(D,e_D)}(L')$ such that $X \simeq iL'$, and this idempotent is unique up to conjugation; the direct image $\beta(i)$ is a primitive idempotent of the algebra $\mathrm{End}_{kN(D,e_D)}(L'(P,e_P))$.

In other words, the mapping $X' \mapsto X'(P,e_P)$ defines a one-to-one correspondence between the isomorphism classes of indecomposable $\mathcal{O}N(D,e_D)$-modules with source triple $(\Delta D, e_D \otimes e_D^{\circ}, V)$ and the isomorphism classes of indecomposable $kN(D,e_D)$-modules with source triple $(\Delta D, e_D \otimes e_D^{\circ}, k)$. This is a special case of the Puig correspondence, which was defined in [Pu-1988a]. The $kN(D,e_D)$-module $kC_H(D)e_D$ is indecomposable with vertex subpair $(\Delta D, e_D \otimes e_D^{\circ})$ and trivial source, so there exists a unique isomorphism class of indecomposable $\mathcal{O}N(D,e_D)$-module M' with source triple $(\Delta D, e_D \otimes e_D^{\circ}, V)$ such that $M'(P,e_P) \simeq kC_H(D)\bar{e}_D$. This proves the existence and uniqueness of the Green correspondent of M', *i.e.*, the $\mathcal{O}(H \times C_H(P))\Delta N_G(P)$-module M. □

4.3 The global case

In the section, we assume that the prime number p is odd. We consider a finite group G, and a block e of the group algebra $\mathcal{O}G$. We assume that the group G admits an e-subpair (P,e_P) and a Sylow e-subpair (D,e_D) such that

- $(P,e_P) \leqslant (D,e_D)$ and $D \leqslant C_G(P)$;
- $N_G(Q,e_Q) = O_{p'}(C_G(Q))\, C_{N_G(Q,e_Q)}(P)$
  for any nontrivial e-subpair $(Q,e_Q) \leqslant (D,e_D)$.

For any nontrivial subgroup Q of D, there is a unique block e_Q of the centraliser $C_G(Q)$ such that $(Q,e_Q) \leqslant (D,e_D)$. We may consider the idempotent e_Q as a block of the local subgroup $N_G(Q,e_Q)$. We denote by $(N_D(Q), e_{N_D(Q)})$ the normaliser subpair of the subgroup Q in the maximal e-subpair (D, e_D). Then $(N_D(Q), e_{N_D(Q)})$ is an e_Q-subpair of the group $N_G(Q, e_Q)$, although it needs not be maximal. We let S_Q be a normal p'-subgroup of $N_G(Q, e_Q)$ such that $S_Q \leqslant C_G(Q)$ and $N_G(Q, e_Q) = S_Q N_{C_G(P)}(Q, e_Q)$. This might be $O_{p'}(C_G(Q))$ or a smaller normal subgroup. We let $\bar{b}_Q$ be an $N_D(Q)$-stable block of the group algebra kS_Q such that the block $\bar{e}_{N_D(Q)}$ of the algebra $kC_G(N_D(Q))$ covers the block $\mathrm{br}_{N_D(Q)}(\bar{b}_Q)$ of the algebra $kC_S(N_D(Q))$. We let $u_Q \in \mathcal{D}(N_D(Q))$ be the class defined by the Dade $N_D(Q)$-algebra $kS_Q\bar{b}_Q$, and $u_Q(P) \in \mathcal{D}(N_D(Q))$ be the class defined by the Brauer quotient $\mathrm{Br}_P(kS_Q\bar{b}_Q) \simeq kS_Q\,\mathrm{br}_P(b_Q)$. We set

$$v_Q \;=\; u_Q - u_Q(P) \;\in\; \mathcal{D}(N_D(Q)),$$

and we let V_Q be an indecomposable capped endopermutation $kN_D(Q)$-module that belongs to the class v_Q.

Lemma 4.13. *Let Q be a subgroup of the defect group D.*

(i) If R is a subgroup of D that normalises Q, then

$$\big[V_Q(R)\big] \;=\; \big[\mathrm{Res}^{N_D(R)}_{N_D(Q,R)} V_R\big] \quad \textit{in the Dade group } \mathcal{D}(N_D(Q,R)).$$

(ii) If g is an element of the group G such that ${}^g(Q, e_Q) \leqslant (D, e_D)$, then

$$\big[\mathrm{Res}^{N_D(Q)}_{N_{D\cap D^g}(Q)} V_Q\big] \;=\; \big[\mathrm{Res}^{N_{D^g}(R)}_{N_{D\cap D^g}(Q)} g^{-1}V_{{}^gQ}\big] \quad \textit{in the Dade group } \mathcal{D}(N_{D\cap D^g}(R)).$$

Proof. Let us fix a non-trivial p-subgroup Q of the defect group D. We set $G_Q = N_G(Q, e_Q)$ and $H_Q = C_G(Q)$. By assumption, we have the factorisation $G_Q = S_Q C_{G_Q}(P)$. Thus all the assumptions of Section 4.2 are satisfied. We choose a maximal e_Q-subpair (D_Q, f_Q) of the group G_Q such that $(N_D(Q), e_{N_D(Q)}) \leqslant (D_Q, f_Q)$. We denote by M_Q the indecomposable Brauer-friendly $k(H_Q \times C_{H_Q}(P))\Delta C_{G_Q}(P)$-module of Theorem 4.7. We denote by W_Q the source of M_Q relative to the vertex subpair $(\Delta D_Q, f_Q \otimes f_Q^{o})$; we consider W_Q as a kD_Q-module. Then it follows from Lemma 4.11 that the $kN_D(Q)$-module V_Q is isomorphic to a direct summand of the restriction $\mathrm{Res}^{D_Q}_{N_D(Q)} W_Q$. Since M_Q is Brauer-friendly, this restriction is uniquely defined by the e_Q-subpair $(N_D(Q), e_{N_D(Q)})$. In particular, the class v_Q is independent of the choice of the subgroup S_Q and of the block b_Q; it only depends on the maximal subpair (D, e_D) and the p-group Q.

We now suppose that Q is a normal subgroup of R. On the one hand, S_Q is a normal p'-subgroup of G_Q such that $S_Q \leqslant H_Q$ and $G_Q = S_Q H_Q$, and $\bar{b}_Q$ is an $N_D(Q)$-stable block of kS_Q such that the block $e_{N_D(Q)}$ covers $\mathrm{br}_{N_D(Q)}(\bar{b}_Q)$. We set $G_{Q,R} = N_G(Q, R, e_R)$, $S_{Q,R} = C_{S_Q}(R)$ and $b_{Q,R} = \mathrm{br}_R(b_Q)$. Then $S_{Q,R}$ is a normal p'-subgroup of $G_{Q,R}$ such that $S_{Q,R} \leqslant H_R$ and $G_{Q,R} = S_{Q,R}C_{G_{Q,R}}(P)$, and $b_{Q,R}$ is an $N_D(Q,R)$-stable block of $S_{Q,R}$ such that the block $e_{N_D(Q,R)}$ covers $\mathrm{br}_{N_D(Q,R)}(b_{Q,R})$. Let $u_{Q,R} \in \mathcal{D}(N_D(Q,R))$ be the class defined by the Dade $N_D(Q,R)$-algebra $kS_{Q,R}\bar{b}_{Q,R}$, and let $V_{Q,R}$ be an indecomposable capped endopermutation $kN_D(Q,R)$-module that belongs to the class $v_{Q,R} = u_{Q,R} - u_{Q,R}(P)$. As above, the $kN_D(Q,R)$-module $V_{Q,R}$ is isomorphic to a direct summand of the restriction of a source $W_{Q,R}$ of the Brauer-friendly module $M_{Q,R}$ with respect to a maximal e_R-subpair that contains $(N_D(Q,R), e_{N_D(Q,R)})$. By construction, we have $[V_{Q,R}] = [V_Q(R)]$ in the Dade group $\mathcal{D}(N_D(Q,R))$.

On the other hand, let S_R be a normal p'-subgroup of G_R such that $S_R \leqslant H_R$ and $G_R = S_R C_{G_R}(P)$, and $\bar{b}_R$ be an $N_D(R)$-stable block of S_R such that the block $\bar{e}_{N_D(R)}$ covers $\mathrm{br}_{N_D(R)}(\bar{b}_R)$. Then S_R is also a normal p'-subgroup of $G_{Q,R}$ such that $S_R \leqslant C_G(R)$ and $G_{Q,R} = S_R C_{G_{Q,R}}(P)$, and b_R is an $N_D(Q,R)$-stable block of S_R such that the block $\bar{e}_{N_D(Q,R)}$ covers $\mathrm{br}_{N_D(Q,R)}(\bar{b}_R)$. Since the class $[V_{Q,R}]$ is independent of the choice of the subgroup $S_{Q,R}$ and of the block $b_{Q,R}$, it follows that $[V_{Q,R}] = [\mathrm{Res}^{N_D(R)}_{N_D(Q,R)} V_R]$, and (i) is proven. The proof of (ii) is essentially the same. □

For the rest of this section, we make the following assumption

Assumption 4.14. There exists an indecomposable endopermutation $\mathcal{O}D$-module V that is fusion-stable in the group G with respect to the subpair (D, e_D) and that satisfies, for any non-trivial subgroup Q of D,

$$\big[V(Q)\big] \;=\; \big[V_Q\big] \quad \text{in the Dade group } \mathcal{D}(N_D(Q)).$$

We expect this assumption to be always true. We do know it to be true in several cases.

Lemma 4.15. *Assumption 4.14 is true*

(i) if the defect group D is abelian ;

(ii) if the defect group D contains an elementary central subgroup of rank 2;

(iii) if e is the principal block of the group G.

Proof. If v is a class of the Dade group $\mathcal{D}(D)$, then, by [Th-2007, Theorem 14.2], there exists a unique isomorphism class of capped indecomposable endopermutation $\mathcal{O}D$-module, with determinant 1, such that $v = [k \otimes_{\mathcal{O}} V]$.

Firstly, we suppose that the defect group D is abelian. Then we have $N_D(Q) = D$ for any subgroup Q of D. Following [Pu-1991], we consider the function μ on the set of non-trivial subgroups of D such that $\sum_{1 \neq R \leqslant Q} \mu(R) = 1$ for any non-trivial subgroup Q of D. We let V be a capped indecomposable endopermutation $\mathcal{O}D$-module with determinant 1 that satisfies

$$\big[k \otimes_{\mathcal{O}} V\big] \;=\; \sum_{1 \neq Q \leqslant D} \mu(Q) v_Q \quad \text{in the Dade group } \mathcal{D}(D).$$

By Lemma 4.13, the family $(V_Q)_{1 \neq Q \leqslant D}$ satisfies the assumptions of [Pu-1991, Proposition 3.6]. Thus the $\mathcal{O}D$-module V is $N_G(D, e_D)$-stable and satisfies

$$\big[V(Q)\big] \;=\; \big[V_Q\big] \quad \text{in the Dade group } \mathcal{D}(N_D(Q))$$

for any non-trivial subgroup Q of D. Since the defect group D is abelian, the normaliser $N_G(D, e_D)$ controls the e-fusion in the group G. Thus the endopermutation $\mathcal{O}D$-module V is fusion-stable in the group G with respect to the subpair (D, e_D).

Secondly, we suppose that the defect group D contains an elementary central subgroup of rank 2. For any subgroup Q of D, the class $u_Q \in \mathcal{D}(N_D(Q))$ contains the source of a simple module for the block algebra $kS_QN_D(Q)\bar{b}_Q$, where $S_QN_D(Q)$ is a p-nilpotent group. Thus we know from [BK-2006, Proposition 4.4] or [Pu-2008, Theorem 7.8] that the class u_Q lies in the torsion part $\mathcal{D}_t(N_D(Q))$ of the Dade group $\mathcal{D}(N_D(Q))$ (notice that the first reference relies on the classification of finite simple groups, whereas the second reference seems to be independent). By [BT-2008, Theorem 1.1], there is an exact sequence

$$0 \;\to\; \mathcal{D}_t(D) \;\to\; \varprojlim_{1 \neq Q \leqslant D} \mathcal{D}_t(N_D(Q)) \;\to\; \tilde{H}^0(\mathcal{A}_{\geqslant 2}(D)) \;\to\; 0.$$

By lemma 4.13, the family $(V_Q)_{1 \neq Q \leqslant D}$ lies in the direct limit $\varprojlim_{1 \neq Q \leqslant D} \mathcal{D}_t(N_D(Q))$ of the above exact sequence. In the above exact sequence, $\mathcal{A}_{\geqslant 2}(D)$ is the poset of elementary abelian subgroups of D of rank at least 2, and $\tilde{H}^0(\mathcal{A}_{\geqslant 2}(D), \mathbb{F}_2)$ is the additive group of locally $\mathbb{F}_2$-valued functions on $\mathcal{A}_{\geqslant 2}(D)$, quotiented by the subgroup of constant functions. We assume that the defect group D contains an elementary central subgroup of rank 2, so the poset $\mathcal{A}_{\geqslant 2}(D)$ is connected. As a consequence, the obstruction group $\tilde{H}^0(\mathcal{A}_{\geqslant 2}(D), \mathbb{F}_2)$ is trivial. Thus there exists a unique class v in the torsion Dade group $\mathcal{D}_t(D)$ such that

$$\big[V(Q)\big] \;=\; \big[V_Q\big] \quad \text{in the Dade group } \mathcal{D}(N_D(Q))$$

for any non-trivial subgroup Q of D, where V is a capped indecomposable endopermutation $\mathcal{O}D$-module with determinant 1 such that the reduction $k \otimes_{\mathcal{O}} V$ belongs to the class v. Let (R, e_R) be a subpair of (D, e_D) and let $g \in G$ be an element such that ${}^g(R, e_R) \leqslant (D, e_D)$. Set $W = \mathrm{Res}^D_R V$ and $W' = \mathrm{Res}^{D^g}_R\, g^{-1}V$. For any non-trivial subgroup Q of R, we have

$$W(Q) \simeq \mathrm{Res}^{N_D(Q)}_{N_R(Q)} V(Q) \quad \text{and} \quad W'(Q) \simeq \mathrm{Res}^{N_{D^g}(Q)}_{N_R(Q)}\, g^{-1}(V({}^gQ)).$$

In terms of classes in the Dade group $\mathcal{D}(N_R(Q))$, Lemma 4.13 (ii) brings

$$\big[W(Q)\big] = \big[\mathrm{Res}^{N_D(Q)}_{N_R(Q)} V_Q\big] = \big[\mathrm{Res}^{N_{D^g}(Q)}_{N_R(Q)}\, g^{-1}V_{{}^gQ}\big] = \big[W'(Q)\big].$$

Then the injectivity of the natural map $\mathcal{D}_t(R) \to \varprojlim_{1 \neq Q \leqslant R} \mathcal{D}_t(N_R(Q))$ implies that $[W] = [W']$ in the Dade group $\mathcal{D}(R)$. It follows that the endopermutation $\mathcal{O}D$-module V is fusion-stable in the group G with respect to the subpair (D, e_D).

Thirdly, we suppose that e is the principal block of the group G. Then the block e_Q covers the principal block b_Q of the p'-group S_Q, and V_Q is the trivial $kN_D(Q)$-module for any non-trivial subgroup Q of the defect group D. Thus we can choose V to be the trivial $\mathcal{O}D$-module. □

For a general defect group D, the obstruction group $\tilde{H}^0(\mathcal{A}_{\geqslant 2}(D), \mathbb{F}_2)$ needs not be trivial. However, we know from the classification of finite simple groups that the Z_p^*-theorem is always true. This implies that Assumption 4.14 is satisfied, at least when the centraliser $C_G(P)$ controls the p-fusion in the group G (and not only the e-fusion). We do hope that a careful study of the direct image of the family $(v_Q)_{1 \neq Q \leqslant D}$ in the obstruction group $\tilde{H}^0(\mathcal{A}_{\geqslant 2}(D), \mathbb{F}_2)$ will show that this direct image is always trivial. This would allow one to prove Theorem 4.1 without any restriction on the defect group D.

We now suppose that Assumption 4.14 is satisfied. By Lemma 4.12, there exists a unique indecomposable Brauer-friendly $\mathcal{O}(G \times C_G(P))$-module with source triple $(\Delta D, e_D \otimes e_D^{\circ}, V)$ such that the slashed module $M(\Delta P, e_P \otimes e_P^{\circ})$ admits the $k(C_G(P) \times C_G(P))$-module $kC_G(P)e_P$ as a direct summand.

Lemma 4.16. *With the above notations and assumptions, let Q be a non-trivial subgroup of the defect group D. Then the slashed module $M(\Delta Q, e_Q \otimes e_{PQ})$ induces a Morita equivalence*

$$kC_G(Q)e_Q \ \sim \ kC_G(PQ)e_{PQ}$$

Proof. For the sake of shortness, when R is any subgroup of the defect group D, we write

$$C(R) = (C_G(R) \times C_G(PR)) \ ; \ N(R) = C(R)\Delta N_G(R, e_R) \ ; \ M(R) = M(\Delta R, e_R \otimes e_{PR}).$$

By Theorem 3.15, the slashed module $M(Q)$ is a $kC(Q)$-module, and it admits a non-unique extension to a $kN(Q)$-module. Then the slashed module $M(Q)(P)$ is automatically a $kN(P,Q)$-module, where $N(P,Q) = C(P) \cap N(Q)$. Similarly, the slashed module $M(P)$ is a $kC(P)$-module. Moreover, the kD-module $V(P)$ is trivial and V is a source of M, so we deduce from Theorem 3.15 (iii) that $M(P)$ is a p-permutation $kC(P)$-module. Thus we can choose the slashed module $M(P)(Q)$ to be the Brauer quotient $\mathrm{Br}_{(\Delta Q, e_{PQ} \otimes e_{PQ}^{\circ})}(M(P))$. This automatically makes $M(P)(Q)$ a $kN(P,Q)$-module. By Lemma 3.16 (i), there is an isomorphism of $kC(PQ)$-modules $M(Q)(P) \simeq M(P)(Q)$. Moreover, we may have chosen the extension of $M(Q)$ to a $kN(Q)$-module in such a way that there is an isomorphism of $kN(P,Q)$-modules $M(Q)(P) \simeq M(P)(Q)$. From now on, we endow the slashed module $M(Q)$ with the unique structure of $kN(Q)$-module that has this property.

By construction of M, we know that the $kC(P)$-module $M(P)$ admits $kC_G(P)e_P$ as a direct summand. As a consequence, the Brauer quotient $M(P)(Q)$ admits the $kN(P,Q)$-module $kC_G(PQ)e_{PQ}$ as a direct summand. The isomorphism $M(Q)(P) \simeq M(P)(Q)$ implies that the $kN(P,Q)$-module $M(Q)(P)$ also admits $kC_G(PQ)e_{PQ}$ as a direct summand. Let M_Q^0 be an indecomposable direct summand of the $kN(Q)$-module $M(Q)$ such that the slashed module $M_Q^0(P)$ admits the $kN(P,Q)$-module $kC_G(PQ)e_{PQ}$ as a direct summand.

Let (R, f) be a maximal e_Q-subpair of the group $N_G(Q, e_Q)$. Then the Brauer quotient $\mathrm{Br}_{(\Delta R, f \otimes f^{\circ})}(kC_G(PQ)e_{PQ}) \simeq kC_G(R)f$ is non-zero. By Lemma 3.16, it follows that the slashed module $M_Q^0(\Delta R, f \otimes f^{\circ})$ is non-zero. By Theorem 3.15 (iii), a vertex subpair of M_Q^0 must be contained in a conjugate of the vertex subpair $(\Delta D, e_D \otimes e_D^{\circ})$ of M. Thus $(\Delta R, f \otimes f^{\circ})$ is a vertex subpair of M_Q^0.

If g is an element of the group G such that ${}^g(Q, e_Q) \leqslant (D, e_D)$, then Lemma 3.16 (ii) and brings an isomorphism of $kC(Q)$-modules

$$M(Q) \simeq (g^{-1}, g^{-1}) \cdot M({}^gQ).$$

This is even an isomorphism of $kN(Q)$-module, thanks to the above choice of extensions of $M(Q)$ to a $kN(Q)$-module and $M({}^gQ)$ to a $kN({}^gQ)$-module. Thus, up to replacing the subpair (Q, e_Q) by a conjugate, we may suppose that the normaliser subpair $(N_D(Q), e_{N_D(Q)})$ of the subpair (Q, e_Q) in (D, e_D) is a maximal e_Q-subpair of the group $N_G(Q, e_Q)$. Then we deduce from Theorem 3.15 (iii) that $(\Delta N_D(Q), e_{N_D(Q)} \otimes e_{N_D(Q)}^{\circ}, V_Q)$ is a source triple of the indecomposable $kN(Q)$-module M_Q^0. Now the unicity in Theorem 4.7 implies that the $kN(Q)$-module M_Q^0 induces an $N_G(Q, e_Q)/C_G(Q)$-equivariant Morita equivalence

$$kC_G(Q)\bar{e}_Q \sim kC_G(PQ)\bar{e}_{PQ}.$$

The next step uses descending induction on the group Q to prove that the $kC(Q)$-module $M(Q)$ is indecomposable, *i.e.*, $M(Q) = M_Q^0$. Firstly, let the $kN(D)$-module M' be a Green correspondent of M with respect to the vertex subpair $(\Delta D, e_D \otimes e_D^{\circ})$. We know from the proof of Lemma 4.12 that the slashed module $M'(P)$ is isomorphic to $kC_G(D)e_D$ as a $kC(D)$-module . So the slashed module $M(D)$ is also isomorphic to $kC_G(D)e_D$ as a $kC(D)$-module. This proves that $M(D) = M_D^0$.

Secondly, let Q be a proper subgroup of D and suppose that $M(R) = M_R^0$ for any p-group R such that $Q < R \leqslant D$. Once again, we may suppose that the normaliser subpair $(N_D(Q), e_{N_D(Q)})$ is a maximal e_Q-subpair of the group $N_G(Q, e_Q)$. We consider a Krull-Schmidt decomposition

$$M(Q) = M_Q^0 \oplus \ldots \oplus M_Q^n,$$

of the $kN(Q)$-module $M(Q)$, and we suppose that $n \geqslant 1$. Let (R, f) be a vertex subpair of the $kN(Q)$-module M_Q^1. We may suppose that (R,f) is contained in the maximal $(e_Q\otimes e_Q^{\circ})$-subpair $((C_D(Q)\times C_D(Q))\Delta N_D(Q), e_{N_D(Q)}\otimes e^{\circ}_{N_D(Q)})$. On the one hand, by Theorem 3.15 (iii), the subpair (R,f) must be contained in a $(G\times C_G(P))$-conjugate of $(\Delta D, e_D\otimes e_D^{\circ})$. Thus we have $\mathrm{Br}_{(R,f)}(kGe)\neq 0$. Since the subpair $(\Delta Q, e_Q\otimes e_Q^{\circ})$ is normalised by (R,f) and $\mathrm{Br}_{(\Delta Q, e_Q\otimes e_Q^{\circ})}(kG\bar{e})\simeq kC_G(Q)\bar{e}_Q$, it follows that $\mathrm{Br}_{(R,f)}(kC_G(Q)\bar{e}_Q)\neq 0$. So the subpair (R,f) is contained in a $(C_G(Q)\times C_G(Q))\Delta N_G(Q,e_Q)$-conjugate of the vertex subpair $(\Delta N_D(Q), e_{N_D(Q)}\otimes e^{\circ}_{N_D(Q)})$ of the indecomposable $k(C_G(Q)\times C_G(Q))\Delta N_G(Q,e_Q)$-module $kC_G(Q)e_Q$. Moreover the subgroup $N(P,Q)$ controls the $(e_Q\otimes e_Q^{\circ})$-fusion in the group $(C_G(Q)\times C_G(Q))\Delta N_G(Q,e_Q)$. So (R,f) is contained in an $N(P,Q)$-conjugate of $(\Delta N_D(Q), e_{N_D(Q)}\otimes e^{\circ}_{N_D(Q)})$. We may choose $(R,f)=(\Delta R', e_{R'}\otimes e^{\circ}_{PR'})$ for some subgroup R' of $N_D(Q)$. If $Q<R'$, then we obtain

$$M(R')\simeq M(Q)(R')=M_Q^0(R')\oplus\ldots\oplus M_Q^n(R'),$$

where at least the direct summands $M_Q^0(R')$ and $M_Q^1(R')$ are non-zero. This contradicts the indecomposability of the $kC(R')$-module $M(R')=M^0_{R'}$. If $Q=R'$, then Lemma 3.23 implies that the $k(G\times C_G(P))$-module M has an indecomposable direct summand with vertex subpair $(\Delta Q, e_Q\otimes e^{\circ}_{PQ})$, another contradiction. So the lemma is proven. □

For the reader's convenience, we quote [Li-2013, Theorem 1.1], which is not published yet. We slightly adapt the notations to fit those of the present chapter.

Theorem (Linckelmann). *Let A, B be (almost) source algebras of blocks of finite group algebras over $\mathcal{O}$ having a common defect group D and the same fusion system $\mathbf{F}$ on D. Let V be an $\mathbf{F}$-stable indecomposable endopermutation $\mathcal{O}D$-module with vertex D, viewed as an $\mathcal{O}\Delta D$-module through the canonical isomorphism $\Delta D\simeq D$. Let M be an indecomposable direct summand of the (A,B)-bimodule*

$$A\otimes_{\mathcal{O}D}\mathrm{Ind}_{\Delta D}^{D\times D}V\otimes_{\mathcal{O}D}B$$

Suppose that $M\otimes_B M^*\neq 0$. Then, for any nontrivial fully $\mathbf{F}$-centralised subgroup Q of D, there is a canonical $(\mathrm{Br}_Q(A),\mathrm{Br}_Q(B))$-bimodule $M(\Delta Q)$ satisfying $\mathrm{End}_k(M(\Delta Q))\simeq \mathrm{Br}_{\Delta Q}(\mathrm{End}_{\mathcal{O}}(M))$. Moreover, if for all nontrivial fully $\mathbf{F}$-centralised subgroups Q of D the bimodule $M(\Delta Q)$ induces a Morita equiv-

alence between $\mathrm{Br}_Q(A)$ *and* $\mathrm{Br}_Q(B)$*, then* M *and its dual* M^* *induce a stable equivalence of Morita type between* A *and* B*.*

We now have all the tools that we need to prove our main result.

Proof of Theorem 4.1. Let $i \in (\mathcal{O}Ge)^P$ be a source idempotent of the block e such that $\bar{e}_D \operatorname{br}_D(i) \neq 0$, and let $i_P \in (\mathcal{O}C_G(P)e_P)^D$ be a source idempotent of the block e_P such that $\bar{e}_D \operatorname{br}_D(i_P) \neq 0$. Set $A = i\mathcal{O}Gi$ and $B = i_P\mathcal{O}C_G(P)i_P$. Then iMi_P is an indecomposable direct summand of the (A, B)-bimodule $A \otimes_{kD} \operatorname{Ind}_{\Delta D}^{D\times D} V \otimes_{kD} B$, where V is an endopermutation $\mathcal{O}D$-module that is fusion-stable for the common fusion system of the source algebras A and B on the defect group D. Moreover, by Lemma 4.16, the slashed module $iMi_P(\Delta Q)$ induces a Morita equivalence $A(\Delta Q) \sim B(\Delta Q)$ for any subgroup Q of the defect group D. Then Linckelmann's theorem asserts that the (A, B)-bimodule iMi_P induces a stable equivalence $A \sim B$. In terms of block algebras, this means exactly that the $(\mathcal{O}Ge, \mathcal{O}C_G(P)e_P)$-bimodule M induces a stable equivalence

$$\mathcal{O}Ge \;\sim\; \mathcal{O}C_G(P)e_P. \qquad \square$$

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