From local to global in block theory

Strong fusion control and sheaves of local Morita equivalences

Erwan Biland

Université Paris 7 - Université Laval

UCLA

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Contents

- Stable equivalences and local Morita equivalences.
- Gluing local p-permutation modules.
- Equivariant Morita equivalences .
- Induction of interior algebras and Morita equivalences.

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We follow the paper :

Erwan Biland, *An application of equivariant Morita theory*, preprint.

Stable equivalences of Morita type

Let k be a finite field of characteristic p.
Let the subgroup H control p-fusion in the group G.
Let $e \in kG$ and $f \in kH$ be blocks, e^o and f^o the principal blocks.

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For a p-subgroup $Q \leqslant H$, write $G_Q = C_G(Q)$, $H_Q = C_H(Q)$.

Theorem (Rouquier). Let M be a p-permutation (kGe^o, kHf^o)-bimodule which is relatively ΔH-projective. For any nontrivial $Q \leqslant H$, suppose that the bimodule $\mathrm{Br}_{\Delta Q}(M)$ provides a Morita equivalence

$$kG_Q e_Q^o \ \sim \ kH_Q f_Q^o.$$

Then M provides a stable equivalence

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Remark. The $(kG_Q e_Q^o, kH_Q f_Q^o)$-bimodule $\mathrm{Br}_{\Delta Q}(M)$ is endowed with an extra action of $\Delta N_H(Q)$. We call this an $N_H(Q)$-equivariant bimodule.

Strong fusion control and principal block

Proposition. Suppose that $G = O_{p'}(G)H$.
Then the principal block algebras kGe^o and kHf^o are naturally isomorphic.
In particular, the bimodule $e^o kGf^o$ provides a Morita equivalence

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Corollary. Suppose that H controls strong fusion in G, *i.e.*

$$\forall 1 \neq Q \leqslant H, \quad N_G(Q) = O_{p'}(G_Q)N_H(Q).$$

Then the bimodule $e^o kGf^o$ provides a stable equivalence between the principal block algebras kGe^o and kHf^o.

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Then the bimodule $e^o kGf^o$ provides a stable equivalence between the principal block algebras kGe^o and kHf^o.

Proof. The bimodule $e^o kGf^o$ is p-permutation and ΔH-projective.
For any $1 \neq Q \leqslant H$, there is a natural isomorphism

$$e^o_Q kG_Q f^o_Q \ \simeq \ \mathrm{Br}_{\Delta Q}(e^o kGf^o).$$

Gluing local Morita equivalences

Theorem (Rouquier). Let $(M_Q)_{1 \neq Q \leqslant H}$ be a *sheaf* of p-permutation $N_H(Q)$-equivariant (kG, kH)-bimodules. Then there is a relatively ΔH-projective p-permutation (kG, kH)-bimodule M, well-defined up to a projective summand, such that

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$$\forall 1 \neq Q \leqslant H,\ M_Q \simeq \mathrm{Br}_{\Delta Q}(M).$$

Moreover, if the M_Q provide Morita equivalences

$$kG_Q e_Q^o \ \sim\ kH_Q f_Q^o$$

for any $1 \neq Q \leqslant H$, then M provides a stable equivalence

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Külshammer and Robinson's theorem

Let P be a p-subgroup of G, and set $H = N_G(P)$.
Let $e \in kG$ be a block, and set $f = \mathrm{br}_P(e) \in kH$.
Assume that a defect group of e contains P. Then f is a block.

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Theorem (Külshammer, Robinson, Dade, Puig). If $G = O_{p'}(G)H$, then the block algebras kGe and kHf are Morita equivalent.

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Theorem (Külshammer, Robinson, Dade, Puig). If $G = O_{p'}(G)H$, then the block algebras kGe and kHf are Morita equivalent.

Proof. Let $S \triangleleft G$ be a p'-subgroup such that $G = SH$.
Let $b \in kS$ and $b_P = \mathrm{br}_P(b) \in kS_P$ be blocks such that e covers b and f covers b_P, *i.e.* $e \in kGb^G$ and $f \in kH(b_P)^H$.

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By Clifford theory, there is a bimodule M which provides a Morita equivalence

$$kGb^G \sim kH(b_P)^H.$$

Hence eMf provides a Morita equivalence $kGe \sim kGf$.

Question

Set $H = C_G(P)$. Suppose that H controls strong fusion in G.
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- define a notion of a sheaf over the Frobenius category of a non-principal block (easy from [Alperin-Broué]);
- put the local modules M_Q obtained from KR's theorem into a sheaf (our contribution);
- find a gluing method for a sheaf of endo-p-permutation modules, *i.e.* a sheaf of p-permutation matrix algebras (???);
- prove that a bimodule M which locally provides Morita equivalences must globally provide a stable equivalence (already done?).

Clifford theory

Let $S \lhd G$ be a p'-subgroup, and $b \in kS$ be a block.
Set $G_b = \mathrm{Stab}_G(b)$ and $b^G = \mathrm{Tr}_{G_b}^G(b)$.

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Set $G_b = \mathrm{Stab}_G(b)$ and $b^G = \mathrm{Tr}_{G_b}^G(b)$.

G_b acts on the matrix algebra kSb

$\rightsquigarrow$ a cocycle $[\gamma_b] \in H^2(G_b/S, k^\times)$;

$\rightsquigarrow$ a central extension $1 \to k^\times \to \tilde{G}_b \to G_b \to 1$.

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Theorem (Dade ?). There is a block $c \in k[k^\times]$ and a bimodule M which provides a Morita equivalence

$$kGb^G \ \sim \ k[\tilde{G}_b/S]c.$$

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Proof. Let V be a natural module for the matrix algebra kSb. The Clifford extension $\tilde{G}_b$ acts naturally on V, and we set

$$M = \mathrm{Ind}_{\Delta \tilde{G}_b}^{G \times [\tilde{G}_b/S]^{\mathrm{op}}}(V).$$

Classical proof of KR's theorem

Let P be a p-subgroup of G, and suppose that $H = C_G(P)$.
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Proof. The Clifford extensions $\tilde{G}_b/S$ and $\tilde{H}_{b_P}/S_P$ are naturally isomorphic.
There is a composition of Morita equivalences

$$kGb^G \sim k[\tilde{G}_b/S]c \simeq k[\tilde{H}_{b_P}/S_P]c \sim kHb_P.$$

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Problem. The construction depends on the choice of S. This is bad for localization purposes, since $G_Q \cap O_{p'}(G) < O_{p'}(G_Q)$ in general.
For a block $e \in kGb^G$, we need to prove that the bimodule eMf is independant of the choice of S.

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iff there is a G-equivariant (A, B)-bimodule M together with isomorphisms of G-equivariant bimodules

$$M \otimes_B M^\vee \;\simeq\; A, \quad M^\vee \otimes_A M \;\simeq\; B,$$

where $M^\vee = \mathrm{Hom}_A(M, A)$.

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Special case. Let A and B be matrix algebras. There is a G-equivariant Morita equivalence $A \sim B$ iff $[\gamma_A] = [\gamma_B]$ in $H^2(G, k^\times)$.

A functorial proof of KR's theorem

Let P be a p-subgroup of G, and set $H = C_G(P)$.
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Proof.
Let $V = kSb$, a natural module for the matrix algebra $E = kSb \otimes (kSb)^{\mathrm{op}}$.
Let W be a natural module for the matrix algebra

$$Br_{[1\times P]}(E) \simeq kSb \otimes (kS_P b_P)^{\mathrm{op}}.$$

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ΔG acts on V, so $\Delta C_G(P) = \Delta H$ acts on the slashed module W.
W provides an H-equivariant Morita equivalence $kSb \ \sim \ kS_P b_P$.

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We set

$$M = \mathrm{Ind}_{(S\times S_P^{\mathrm{op}})\Delta H}^{G\times H^{\mathrm{op}}}(W).$$

M provides a Morita equivalence $kGb^G \ \sim \ kH(b_P)^H$.

A functorial proof of KR's theorem (equivariant version)

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Proof.
$\Delta(G \rtimes A)$ acts on V, so $\Delta(H \rtimes A)$ acts on the slashed module W.
This makes M an A-equivariant $(kGb^G, kH(b_P)^H)$-bimodule.

Consequences of the new proof

Let P be a p-subgroup of G, and set $H = C_G(P)$.
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For an e-subpair (Q, e_Q), let $E_{(Q,e_Q)}$ be the matrix algebra such that a natural module $M_{(Q,e_Q)}$ for $E_{(Q,e_Q)}$ provides an $N_A(Q, e_Q)$-equivariant Morita equivalence

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There is a natural isomorphism

$$E_{(Q,e_Q)} \;\simeq\; e_Q \mathrm{Br}_{\Delta Q}(E) e_Q\,.$$

A coherent system of equivariant Morita equivalences

Assume that $H = C_G(P)$ controls strong fusion in G.
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Let $e \in kG$ and $f = \mathrm{br}_P(e) \in kH$ be blocks.

Theorem. For each nontrivial e-subpair (Q, e_Q), there is an $(G_Q \otimes H_Q^{\mathrm{op}})\Delta N_H(Q, e_Q)$-interior matrix algebra $E_{(Q,e_Q)}$ such that a natural module $M_{(Q,e_Q)}$ for $E_{(Q,e_Q)}$ provides an $N_H(Q, e_Q)$-equivariant Morita equivalence

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For each normal map $\varphi : (Q, e_Q) \to (R, e_R)$, there is an isomorphism

$$E_{(R,e_R)} \simeq e_R \mathrm{Br}_R \circ \varphi_*(E_Q) e_R.$$

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Theorem. For each nontrivial e-subpair (Q, e_Q), there is an $(G_Q \otimes H_Q^{\mathrm{op}})\Delta N_H(Q, e_Q)$-interior matrix algebra $E_{(Q,e_Q)}$ such that a natural module $M_{(Q,e_Q)}$ for $E_{(Q,e_Q)}$ provides an $N_H(Q, e_Q)$-equivariant Morita equivalence

$$kG_Q e_Q \;\sim\; kH_Q f_Q.$$

For each normal map $\varphi : (Q, e_Q) \to (R, e_R)$, there is an isomorphism

$$E_{(R,e_R)} \simeq e_R \mathrm{Br}_R \circ \varphi_*(E_Q) e_R.$$

This data provide a sheaf on the (punctured) Frobenius category of the block e.

Gluing ?

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To be continued ?