Blocs de Brauer et structure locale des groupes finis

Brauer blocks and local structure in finite groups

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General framework

Field of study : finite groups
No classical groups or reductive algebraic groups here...

Goal : understanding how a group G acts on its p-subgroups (local structure) and deriving global properties

Tools : characters and representations
- over $\mathbb{C}$ or a field of characteristic 0 (ordinary characters)
- over a field of positive caracteristic p (modular characters)

How to use them : Brauer theory
- characters of G can be gathered in p-blocks
- Brauer's three main theorems connect p-blocks of G and p-blocks of subgroups controlling the local structure of G

Fusion : p-elements

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In general, $x^S \neq x^G \cap S$; we have :

$$x^G \cap S = x^S \sqcup x_1^S \sqcup \ldots \sqcup x_k^S.$$

This is fusion : two elements of S may be conjugate in G but not in S.

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Definition. Let H be a subgroup of G. We say that H controls p-fusion in G if, for any such P, Q, g, there exists $h \in H$ such that g and h induce the same morphism :

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Remark. This means that $g = ch$, with $c \in C_G(P)$ and $h \in H$.

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Theorem (Alperin, 1967). H controls p-fusion in G if, and only if, H contains an S_p-subgroup S of G and, for any p-subgroup P of S,

$$N_G(P) \subset C_G(P).H$$

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Example. Let $O_{p'}(G)$ be the maximal normal p'-subgroup in G. Then :

$$G = O_{p'}(G).H \quad \Rightarrow \quad H \text{ controls } p\text{-fusion in } G$$

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The converse is not always true, but...

Two theorems about fusion

Theorem (Frobenius, 1905). Let G be a finite group, p a prime and S a p-Sylow in G.

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Theorem (Glauberman, 1966). Let G be a finite group, and $x \in G$ an element of order 2.

$$\begin{aligned} C_G(x) \text{ controls 2-fusion in } G \quad &\Rightarrow \quad G = O_{2'}(G).C_G(x) \\ &\Rightarrow \quad x \in Z(G \ mod \ O_{2'}(G)) \end{aligned}$$

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A sketch of the proofs will be given later.

Characters

Let $\mathbb{K}$ be a field of characteristic 0.
An ordinary representation of G is an algebra morphism

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Facts. Let χ be an irreducible character of G, and $g \in G$.

$$g \in \ker\chi \quad \Leftrightarrow \quad \chi(g) = \chi(1)$$

$$g \in Z(G \ mod \ \ker\chi) \quad \Leftrightarrow \quad \exists n \in \mathbb{N}^*,\ \chi(g)^n = \chi(1)^n$$

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With Alperin's theorem, we have :

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This will prove the theorem : by induction, we have $H = O_{p'}(H).(S \cap H)$. Since $O_{p'}(H) \leqslant O_{p'}(G)$ ans $G = H.S$, we get $G = O_{p'}(G).S$.

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> Let $g \in G$ map to a p'-element of G/H. Then $g_p \in H$, so $\chi(g) = \chi(g_p) = 1$.
> This means $g \in \ker \chi = H$, so it maps to 1 in G/H.
> Now G/H has no nontrivial p'-element : it is a p-group.

Brauer Blocks

Let G be a finite group and p a prime.
Denote by $\mathbb{K}$ a finite extension of $\mathbb{Q}_p$, $\mathcal{O}$ the ring of algebraic integers in K, and k the residue field, of characteristic p.

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Ordinary characters χ distribute among the p-blocks B.

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Fact. If $G/O_{p'}(G)$ is a p-group S (as in Frobenius theorem), then the characters of the principal block of G are exactly those which factor through S. More precisely, $B_0(G) \simeq \mathcal{O}S$.

Brauer theory

Brauer map. If P is a p-group in G and $PC_G(P) \leqslant H \leqslant N_G(P)$, then a block of $\hat{B}$ of H can be lifted to a block $\hat{B}^G$ of G.

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Brauer's second main theorem. Let χ be a character of G belonging to the block B, s a p-element in G, $P = \langle s \rangle$, and $H = C_G(s)$. If $r \in C_G(s)$ is a p'-element, then :

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In particular, if $B = B_0(G)$ and $r \in O_{p'}(H)$, then all φ belong to $B_0(H)$, so that $r \in \ker \varphi$ and :

$$\chi(sr) = \sum_{\varphi \in B_0(H)} a_\varphi \varphi(s)$$

Very useful for computing characters !

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Let $\mathcal{S}_x = \sum_{x' \in x^G} x' \in Z(\mathcal{O}G)$. We get $\chi(\mathcal{S}_x\mathcal{S}_y) = |x^G|.|y^G|.\chi(xy)$.

Now $\theta : a \mapsto \chi(a)/\chi(1)$ is an algebra morphism, so $\theta(\mathcal{S}_x\mathcal{S}_y) = \theta(\mathcal{S}_x)\theta(mS_y)$.

We get $\chi(xy)\chi(1) = \chi(x)\chi(y)$. Replace y with xy : $\chi(y)\chi(1) = \chi(x)\chi(xy)$.

This proves $\chi(y) = 0$ or $\chi(x)^2 = \chi(1)^2$.

Glauberman's theorem

Theorem (Glauberman, 1966). Let G be a finite group, and $x \in G$ an involution. Suppose $C_G(x)$ controls 2-fusion in G. Then

$$x \in Z(G \ mod \ O_{2'}(G))$$

Take a minimal counter-example. Let S be an S_2-subgroup containing x.

Lemma 1. S also contains an involution $y \neq x$.

Lemma 2. Let χ be an irreducible character in the principal block of G. For any g, h in G, $\chi(x^g y^h) = \chi(xy)$.

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Since $O_{2'}(G) = \bigcap_{\chi \in B_0(G)} \ker \chi$, this is very close to the conclusion...

Glauberman's theorem : proof of lemma 2

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Proof. First consider the case where x^g and y^h commute.

$C_G(x)$ controls fusion so contains the 2-Sylow S. So x and y commute.
Now one proves that $x^g y^h$ is conjugate to xy.

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Let $H = C_G(s)$. Prove that $r \in O_{2'}(H)$. By Brauer's theorems,

$$\chi(x^g y^h) = \chi(sr) = \sum_{\varphi \in B_0(H)} a_\varphi \varphi(sr) = \sum_{\varphi \in B_0(H)} a_\varphi \varphi(s) = \chi(s)$$

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This proves the lemma.

Prospects

Z_p^* Theorem. For G a finite group, p a prime and x an element of order p,

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This can be proved using the classification of finite simple groups.

An independent proof would be of great theoretical interest. It probably requires deeper study of the principal block $B_0(G)$.

The end

This presentation can be found on my home page :

`http://erwanbiland.fr/index.php?page=recherche`

See also a wider presentation by Michel Broué :

`http://www.math.jussieu.fr/~broue/GainesHist2007.pdf`