Brauer blocks and Morita equivalences

Blocs de Brauer et équivalences de Morita

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Modular representations

- G a finite group
- k a field "big enough" for G, with $\mathrm{char}(k) = p > 0$
- $kG = \left\{ \sum_{g \in G} a_g . g \;\middle|\; a_g \in k \right\}$ the group algebra
- ${}_{kG}\mathrm{mod}$ the category of kG-modules

Blocks and idempotents

- $kG = B_1 \oplus \ldots \oplus B_n$

  with B_i indecomposable two-sided ideal

- $1 = e_1 + \ldots + e_n$

  with e_i primitive central idempotent

- ${}_{kG}\mathrm{mod} \simeq {}_{B_1}\mathrm{mod} \oplus \ldots \oplus {}_{B_n}\mathrm{mod}$

  with ${}_{B_i}\mathrm{mod}$ indecomposable fully abelian subcategory

Examples of idempotents and blocks

1. $p \not| \ |G| \quad \Rightarrow \quad kG$ is semi-simple (Maschke)

   Idempotent $e(\chi)$ associated to a character $\chi \in \mathrm{Irr}(G)$

   $$\begin{aligned} kG &= kGe(\chi_1) \oplus \ldots \oplus kGe(\chi_n) \\ &\simeq \mathrm{Mat}_{d_1}(k) \oplus \ldots \oplus \mathrm{Mat}_{d_n}(k) \end{aligned}$$

2. $S = O_{p'}(G)$: biggest normal subgroup with $p \not| \ |S|$

   G acts on $\mathrm{Irr}(S)$: let χ be fixed

   $\rightsquigarrow \quad kGe(\chi)$ is a **sum of blocks** of kG

Classification of blocks

(from stronger to weaker)

1. Morita equivalence : ${}_{B_1}\mathrm{mod} \simeq {}_{B_2}\mathrm{mod}$

   $\rightsquigarrow \quad M \in {}_{B_1}\mathrm{mod}_{B_2}$ such that

   $M \otimes_{B_2} - : {}_{B_2}\mathrm{mod} \to {}_{B_1}\mathrm{mod}$ is an equivalence of categories

2. Derived equivalence : $D({}_{B_1}\mathrm{mod}) \simeq D({}_{B_2}\mathrm{mod})$

   with a stronger version : splendid equivalence (Rickard)

3. Stable equivalence : $\mathrm{Stab}({}_{B_2}\mathrm{mod}) \simeq \mathrm{Stab}({}_{B_2}\mathrm{mod})$

Examples of Morita equivalences

1. $p \not| \ |G|$: blocks of defect zero

$$kGe(\chi) \quad \sim_{\mathrm{Morita}} \quad k$$

2. $G = S \rtimes P$ with S a p'-group and P a p-group : nilpotent blocks

   $e(\chi)$ associated to $\chi \in \mathrm{Irr}(S)$ fixed by G

$$kGe(\chi) \quad \sim_{\mathrm{Morita}} \quad kP$$

3. $G/S = \bar{G}$ with S a p'-group (useful for p-solvable groups)

$$kGe(\chi) \quad \sim_{\mathrm{Morita}} \quad kG^o_\chi e(\lambda)$$

with G^o_χ central extension of $\bar{G}$ by $k^\times$, and $\lambda \in \mathrm{Irr}(k^\times)$

My problem

- $G = S.C_G(P)$, with S a normal p'-group and P a p-group

  What are the relations between the blocks of kG and $kC_G(P)$?

- An application of Glauberman's correspondence :

$$\begin{array}{rccc} & \mathrm{Irr}(S)^P & \leftrightsquigarrow & \mathrm{Irr}(C_S(P)) \\ \zeta \in & \mathrm{Irr}(S)^G & \leftrightsquigarrow & \mathrm{Irr}(C_S(P))^{C_G(P)} \ni \xi \\ & kGe(\zeta) & \sim_{\mathrm{Morita}} & kC_G(P)e(\xi) \end{array}$$

- For some central idempotent e_0 (naturally defined) :

$$kGe_0 \quad \sim_{\mathrm{Morita}} \quad kC_G(P)$$

  What is the part played by the Brauer functor and morphism in this ?